Script **generated by TTT**

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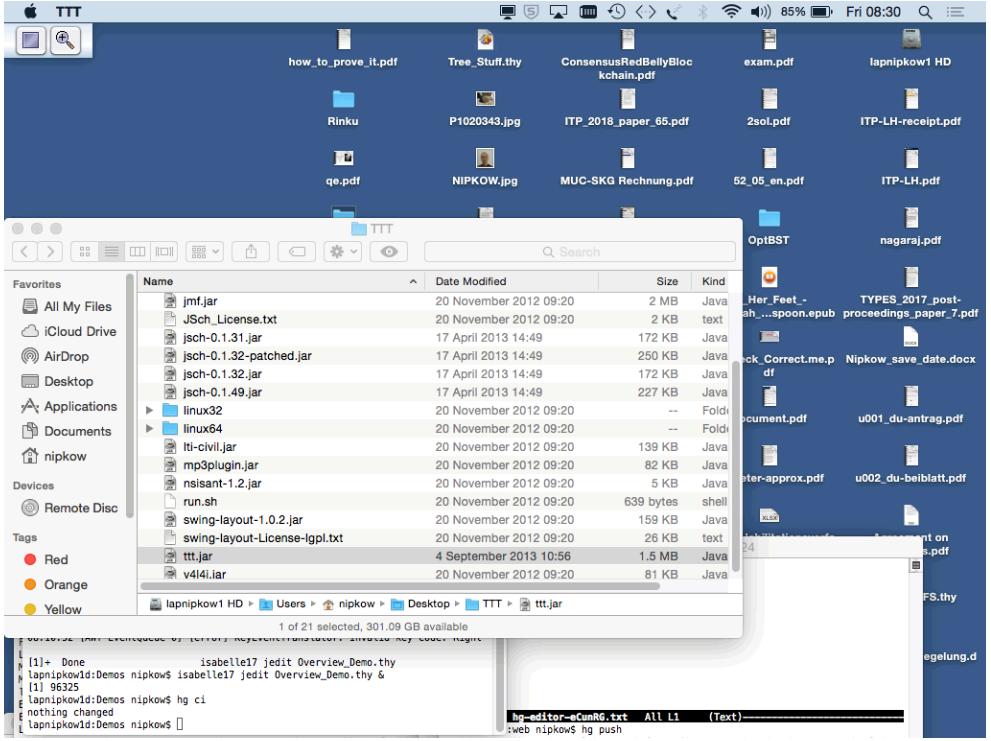

Chapter 1

Introduction

What the course is about

Data Structures and Algorithms
for Functional Programming Languages

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The code is not enough!

Formal Correctness and Complexity Proofs
with the Proof Assistant *Isabelle*

Proof Assistants

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Time consuming

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Government health warnings:

Time consuming
Potentially addictive
Undermines your naive trust in informal proofs

Terminology

Formal = machine-checked
Verification = formal correctness proof

Two landmark verifications

C compiler

Two landmark verifications

C compiler
Competitive with `gcc -O1`

Xavier Leroy
INRIA Paris
using Coq

Operating system
microkernel (L4)

Gerwin Klein (& Co)
NICTA Sydney
using Isabelle

Overview of course

- Week 1–5: Introduction to Isabelle

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- Rest of semester: Search trees, priority queues, etc and their (amortized) complexity

What we expect from you

Functional programming experience with an ML/Haskell-like language

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Functional programming experience with an ML/Haskell-like language

First course in data structures and algorithms

First course in discrete mathematics

You will not survive this course without doing the time-consuming homework

Part I

Isabelle

Chapter 2

Programming and Proving

Notation

Implication associates to the right:

$$A \Longrightarrow B \Longrightarrow C \quad \text{means} \quad A \Longrightarrow (B \Longrightarrow C)$$

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$$\frac{A_1 \quad \dots \quad A_n}{B} \quad \text{means} \quad A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

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HOL = Functional Programming + Logic

HOL has

- datatypes
- recursive functions
- logical operators

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- For the moment: only $term = term$

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- For the moment: only $term = term$, e.g. $1 + 2 = 4$

Types

Basic syntax:

τ	::=	(τ)	
	$\mid$	$bool \mid nat \mid int \mid \ldots$	base types
	$\mid$	$'a \mid 'b \mid \ldots$	type variables

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	$\mid$	$\tau\ set$	sets

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λ-calculus

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$$\frac{t :: \tau_1 \Rightarrow \tau_2 \qquad u :: \tau_1}{t\ u :: \tau_2}$$

Type inference

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User can help with *type annotations* inside the term.
Example: $f\ (x{::}nat)$

Currying

Thou shalt Curry your functions

- Curried: $f :: \tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
- Tupled: $f' :: \tau_1 \times \tau_2 \Rightarrow \tau$

Predefined syntactic sugar

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Prefix binds more strongly than infix:

! $f\ x + y \ \equiv\ (f\ x) + y \ \not\equiv\ f\ (x + y)$ **!**

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- *Mixfix:* *if _ then _ else _*, *case _ of*, ...

Prefix binds more strongly than infix:

! $f\ x + y \ \equiv\ (f\ x) + y \ \not\equiv\ f\ (x + y)$ **!**

Enclose *if* and *case* in parentheses:

! $(if\ \_\ then\ \_\ else\ \_)$ **!**

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imports T1 ... Tn
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(definitions, theorems, proofs, ...)*
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Usually: **imports** `Main`

Concrete syntax

In `.thy` files:
Types, terms and formulas need to be inclosed in "

isabelle jedit

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- Based on *jEdit* editor
- Processes Isabelle text automatically when editing `.thy` files

Overview_Demo.thy

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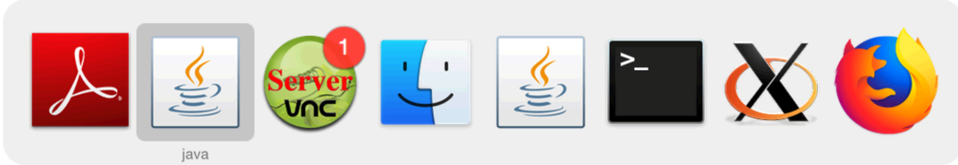

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if-and-only-if: $=$

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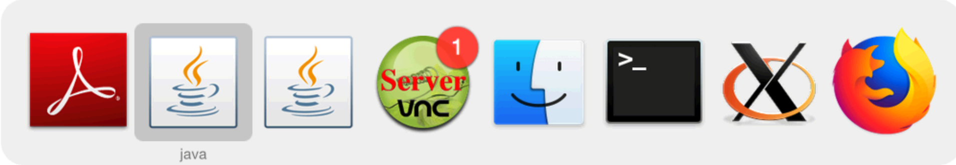

An informal proof

Lemma $add\ m\ 0 = m$
Proof by induction on m.

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  The proof is as follows:
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  The proof is as follows:

$$\begin{aligned} add\ (Suc\ m)\ 0 &= Suc\ (add\ m\ 0) && \text{by def. of } add \\ &= Suc\ m && \text{by IH} \end{aligned}$$

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Lists of elements of type $'a$

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Some lists: $Nil,\ \ Cons\ 1\ Nil,\ \ Cons\ 1\ (Cons\ 2\ Nil),\ \ldots$

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Syntactic sugar:

- $[] = Nil$: empty list

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- $x\ \#\ xs = Cons\ x\ xs$:
  list with first element x (*"head"*) and rest xs (*"tail"*)

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  list with first element x (*"head"*) and rest xs (*"tail"*)
- $[x_1, \ldots, x_n] = x_1\ \#\ \ldots\ x_n\ \#\ []$

Structural Induction for lists

To prove that $P(xs)$ for all lists xs, prove

- $P([])$ and
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$$\frac{P([]) \qquad \bigwedge x\ xs.\ P(xs) \Longrightarrow P(x\#xs)}{P(xs)}$$

`List_Demo.thy`

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