Script **generated by TTT**

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General recursion: Quicksort

Example

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quicksort :: Ord a => [a] -> [a]
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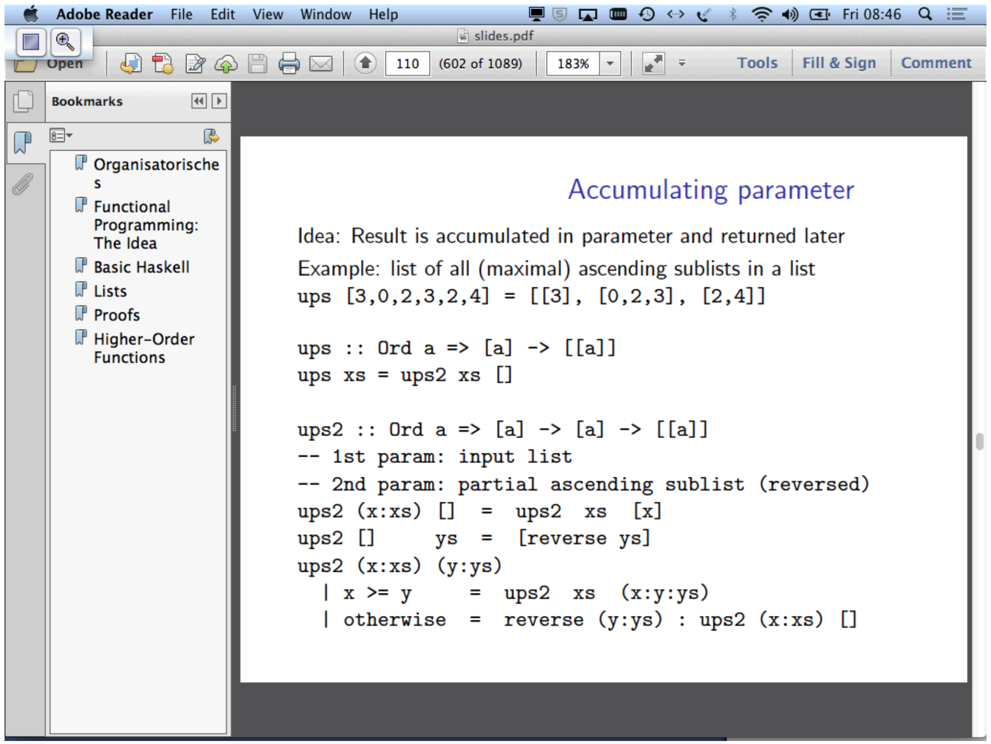
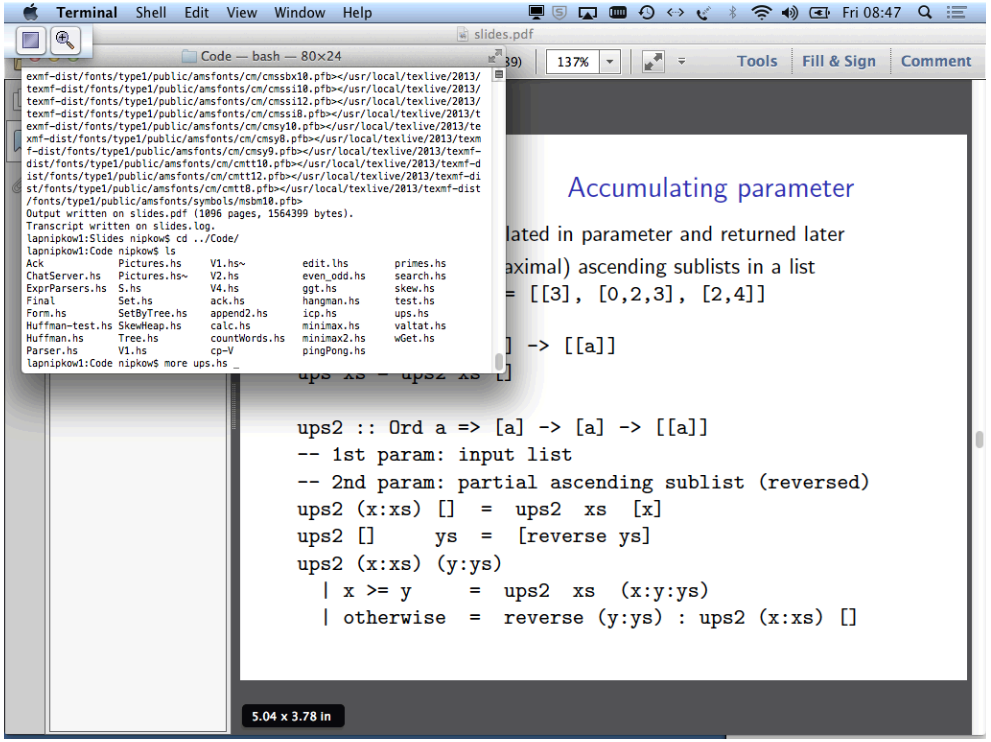
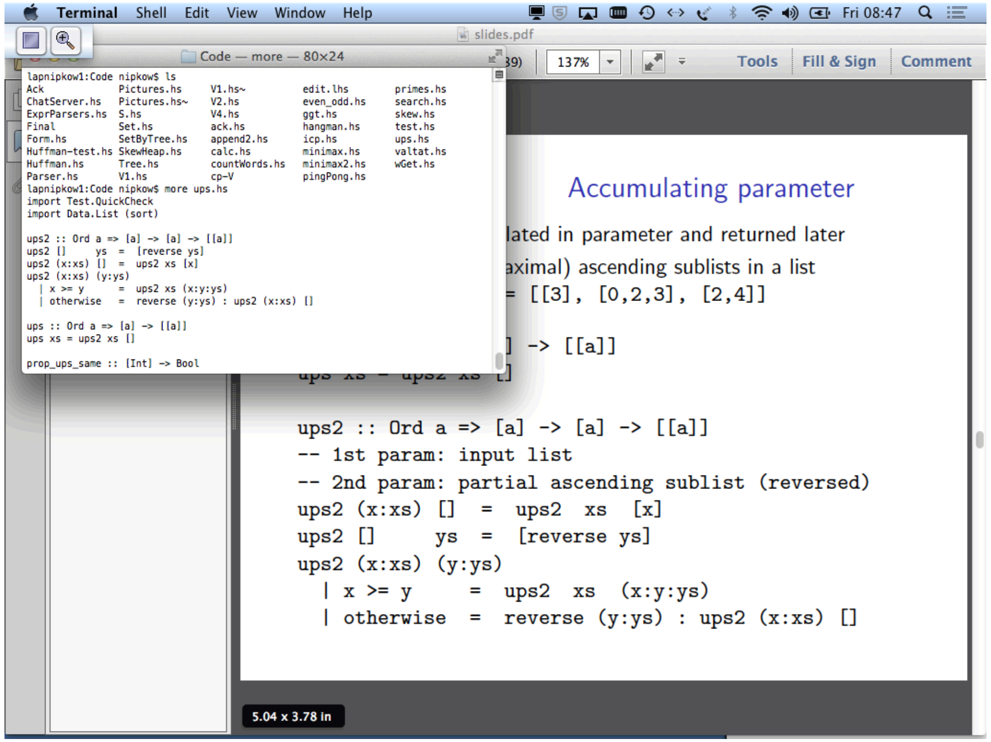
  | otherwise   =  reverse (y:ys) : ups2 (x:xs) []
```

How can we quickCheck the result of ups?







How can we `quickCheck` the result of `ups`?

Convention

Identifiers of list type end in 's':

`xs`, `ys`, `zs`, ...

Mutual recursion

Example

```
even :: Int -> Bool
even n  =  n == 0 || n > 0 && odd (n-1) || odd (n+1)

odd :: Int -> Bool
odd n  =  n /= 0 && (n > 0 && even (n-1) || even (n+1))
```

Scoping by example

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x = y + 5
y = x + 1 where x = 7
f y = y + x

> f 3
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- Order of definitions is irrelevant
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Zeigen Sie, dass TUM TOP ist!

5. Proofs

Aim

Guarentee functional (I/O) properties of software

- Testing can guarantee properties for some inputs.
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QuickCheck is good, proof is better

5.1 Proving properties

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What do we prove?

Equations `e1 = e2`

A first, simple example

Remember:

```
       [] ++ ys = ys
   (x:xs) ++ ys = x : (xs ++ ys)
```

Proof of `[1,2] ++ [] = [1] ++ [2]`:

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Observation: first used equations from left to right (ok),

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Proofs of `e1 = e2` are often better presented as two reductions to some expression `e`:

```
e1 = ...  = e
e2 = ...  = e
```

Fact If an equation does not contain any variables, it can be proved by evaluating both sides separately and checking that the result is identical.

Properties of recursive functions are proved by induction

Induction on natural numbers: see Diskrete Strukturen

Induction on lists: here and now

Structural induction on lists

To prove property `P(xs)` for all finite lists `xs`

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Base case: Prove `P([])` and

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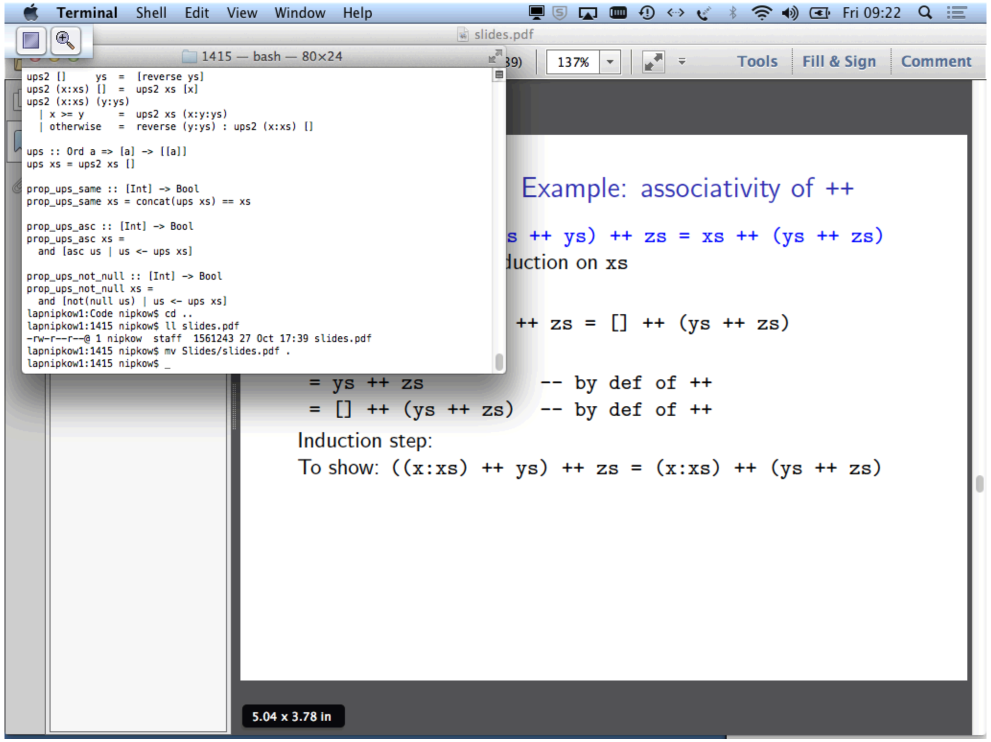

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It is a special case of induction on the length of xs.

Example: associativity of ++

Lemma app_assoc: `(xs ++ ys) ++ zs = xs ++ (ys ++ zs)`
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Lemma app_assoc: `(xs ++ ys) ++ zs = xs ++ (ys ++ zs)`
Proof by structural induction on `xs`

Base case:
To show: `([] ++ ys) ++ zs = [] ++ (ys ++ zs)`

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= ys ++ zs             -- by def of ++
= [] ++ (ys ++ zs)     -- by def of ++
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Induction step:
IH: `([] ++ ys) ++ zs = [] ++ (ys ++ zs)`
To show: `((x:xs) ++ ys) ++ zs = (x:xs) ++ (ys ++ zs)`

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((x:xs) ++ ys) ++ zs
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Induction template

Lemma `P(xs)`
Proof by structural induction on `xs`

Base case:
To show: `P([])`

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To show: `length ([] ++ ys) = length [] + length ys`

```
 length ([] ++ ys)
 = length ys              -- by def of ++
 length [] + length ys
 = 0 + length ys          -- by def of length
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= 1 + length(xs ++ ys)       -- by def of length
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Lemma app_Nil2: `xs ++ [] = xs`
Proof exercise

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= reverse ys ++ (reverse xs ++ [x])    -- by Lemma app_assoc
reverse ys ++ reverse(x:xs)
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= (reverse ys ++ reverse xs) ++ [x]  -- by IH
= reverse ys ++ (reverse xs ++ [x])  -- by Lemma app_assoc
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reverse ys ++ reverse(x:xs)
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Two further tricks

- Proof by cases
- Generalization

Example: proof by cases

`rem x []  =  []`
`rem x (y:ys) | x==y        =  rem x ys`
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Lemma `rem z (xs ++ ys) = rem z xs ++ rem z ys`

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Proof by cases:

Case `z == x`:

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 rem z ((x:xs) ++ ys)
 = rem z (xs ++ ys)          -- by def of ++ and rem
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rem x []     = []
rem x (y:ys) | x==y      = rem x ys
             | otherwise = y : rem x ys
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rem z (x:xs) ++ rem z ys
= rem z xs ++ rem z ys     -- by def of rem
```

Case `z /= x`:

```
rem z ((x:xs) ++ ys)
= x : rem z (xs ++ ys)         -- by def of ++ and rem
= x : (rem z xs ++ rem z ys)   -- by IH
rem z (x:xs) ++ rem z ys
```

Proof by cases

Works just as well for `if-then-else`,

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Works just as well for `if-then-else`, for example

```
rem x []      =  []
rem x (y:ys)  =  if x == y then rem x ys
                 else y : rem x ys
```

Inefficiency of `reverse`

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reverse [1,2,3]
```

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