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12. Lazy evaluation

Introduction

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lazy evaluation („verzögerte Auswertung")

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- Avoids unnecessary evaluations
- Terminates as often as possible

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lazy evaluation („verzögerte Auswertung")

Advantages:

- Avoids unnecessary evaluations
- Terminates as often as possible
- Supports infinite lists
- Increases modularity

Therefore Haskell is called a *lazy functional language*.

Evaluating expressions

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Expressions are evaluated (*reduced*) by successively applying definitions until no further reduction is possible.

Example:

```
sq :: Integer -> Integer
sq n  =  n * n
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sq(3+4)
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Any two terminating evaluations of the same Haskell expression lead to the same final result.

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Example
Let n have value 0 initially.

Two evaluations:

```
n + (n := 1)  =  0 + (n := 1)  =  0 + 1  =  1
n + (n := 1)  =  n + 1  =  1 + 1  =  2
```

Reduction strategies

An expression may have many reducible subexpressions:

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```
sq (3+4)  =  (3+4) * (3+4)
```

Comparison: termination

Definition:

```
loop = tail loop
```

Innermost reduction:

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Innermost reduction:


```
fst (1,loop)  =  fst(1,tail loop)
              =  fst(1,tail(tail loop))
              =  ...
```

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Theorem If expression e has a terminating reduction sequence, then outermost reduction of e also terminates.

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Outermost reduction:

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Theorem If expression e has a terminating reduction sequence, then outermost reduction of e also terminates.

Outermost reduction terminates as often as possible

Why is this useful?

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Example

Can build your own control constructs:

```
switch :: Int -> a -> a -> a
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switch :: Int -> a -> a -> a
switch n x y
  | n > 0      =  x
  | otherwise  =  y
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Can build your own control constructs:

```
switch :: Int -> a -> a -> a
switch n x y
  | n > 0      =  x
  | otherwise  =  y

fac :: Int -> Int
fac n = switch n (n * fac(n-1)) 1
```

Comparison: Number of steps

Innermost reduction:

```
sq (3+4) =  sq 7  =  7 * 7  =  49
```

Comparison: Number of steps

Innermost reduction:

```
sq (3+4) = sq 7 = 7 * 7 = 49
```

Outermost reduction:

```
sq(3+4) = (3+4)*(3+4) = 7*(3+4) = 7*7 = 49
```

```
sq(3+4)
```

```
sq(3+4)  =  •  *  •   =   •  *  •
             ↘   ↙         ↘   ↙
              3+4            7
```

```
sq(3+4)  =  •  *  •   =   •  *  •   =  49
             ↘   ↙         ↘   ↙
              3+4            7
```

```
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The expression 3+4 is only evaluated *once!*


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           3+4        7
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Lazy evaluation := outermost reduction + sharing

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- Arguments are not necessarily evaluated fully,
  but only far enough to evaluate the function.
  (Remember `fst (1,loop)`)

The principles of lazy evaluation:

- Arguments of functions are evaluated only
  if needed to continue the evaluation of the function.
- Arguments are not necessarily evaluated fully,
  but only far enough to evaluate the function.
  (Remember `fst (1,loop)`)
- Each argument is evaluated at most once (sharing!)

Pattern matching

Example

```
f :: [Int] -> [Int] -> Int
f []     ys     =  0
f (x:xs) []     =  0
f (x:xs) (y:ys) =  x+y
```

Pattern matching

Example

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f :: [Int] -> [Int] -> Int
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Lazy evaluation:

`f [1..3] [7..9]`

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Lazy evaluation:

```
f [1..3] [7..9]                    -- does f.1 match?
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f :: [Int] -> [Int] -> Int
f []      ys     =  0
f (x:xs)  []     =  0
f (x:xs) (y:ys)  =  x+y
```

Lazy evaluation:

```
f [1..3] [7..9]                    -- does f.1 match?
=  f (1 : [2..3]) [7..9]
```


Pattern matching

Example

```
f :: [Int] -> [Int] -> Int
f []      ys     =  0
f (x:xs)  []     =  0
f (x:xs) (y:ys)  =  x+y
```

Lazy evaluation:

```
f [1..3] [7..9]                    -- does f.1 match?
=  f (1 : [2..3]) [7..9]           -- does f.2 match?
```


Pattern matching

Example

```
f :: [Int] -> [Int] -> Int
f []      ys     =  0
f (x:xs)  []     =  0
f (x:xs) (y:ys)  =  x+y
```

Lazy evaluation:

```
f [1..3] [7..9]                    -- does f.1 match?
=  f (1 : [2..3]) [7..9]           -- does f.2 match?
=  f (1 : [2..3]) (7 : [8..9])
```

Pattern matching

Example

```
f :: [Int] -> [Int] -> Int
f []      ys     =  0
f (x:xs) []      =  0
f (x:xs) (y:ys) =  x+y
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Lazy evaluation:

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f [1..3] [7..9]                  -- does f.1 match?
=  f (1 : [2..3]) [7..9]         -- does f.2 match?
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Example

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f [1..3] [7..9]                  -- does f.1 match?
=  f (1 : [2..3]) [7..9]         -- does f.2 match?
=  f (1 : [2..3]) (7 : [8..9])   -- does f.3 match?
```


```
=  1+7
=  8
```

Guards

Example

```
f m n p | m >= n && m >= p  =  m
        | n >= m && n >= p  =  n
        | otherwise         =  p
```

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f m n p | m >= n && m >= p  =  m
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Lazy evaluation:

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f (2+3) (4-1) (3+9)
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Example

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f m n p | m >= n && m >= p  =  m
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Lazy evaluation:

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f (2+3) (4-1) (3+9)
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Lazy evaluation:

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f (2+3) (4-1) (3+9)
  ?  2+3 >= 4-1 && 2+3 >= 3+9
```


```
  ?  =  5 >= 3 && 5 >= 3+9
  ?  =  True    && 5 >= 3+9
  ?  =  5 >= 3+9
```

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Example

```
f m n p | m >= n && m >= p  =  m
        | n >= m && n >= p  =  n
        | otherwise         =  p
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Lazy evaluation:

```
f (2+3) (4-1) (3+9)
  ?  2+3 >= 4-1 && 2+3 >= 3+9
  ?  =  5 >= 3 && 5 >= 3+9
  ?  =  True    && 5 >= 3+9
  ?  =  5 >= 3+9
```


```
  ?  =  5 >= 12
  ?  =  False
```

Guards

Example

```
f m n p | m >= n && m >= p  =  m
        | n >= m && n >= p  =  n
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Lazy evaluation:

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f (2+3) (4-1) (3+9)
  ?  2+3 >= 4-1 && 2+3 >= 3+9
  ?  =  5 >= 3 && 5 >= 3+9
  ?  =  True    && 5 >= 3+9
  ?  =  5 >= 3+9
  ?  =  5 >= 12
  ?  =  False
```


```
  ?  3 >= 5 && 3 >= 12
```

Guards

Example

```
f m n p | m >= n && m >= p  =  m
        | n >= m && n >= p  =  n
        | otherwise         =  p
```

Lazy evaluation:

```
f (2+3) (4-1) (3+9)
  ?  2+3 >= 4-1 && 2+3 >= 3+9
  ?  =  5 >= 3 && 5 >= 3+9
  ?  =  True   && 5 >= 3+9
  ?  =  5 >= 3+9
  ?  =  5 >= 12
  ?  =  False
  ?  3 >= 5 && 3 >= 12
  ?  =  False && 3 >= 12
```

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f m n p | m >= n && m >= p  =  m
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Lazy evaluation:

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f (2+3) (4-1) (3+9)
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  ?  =  5 >= 3+9
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  ?  =  False
  ?  3 >= 5 && 3 >= 12
  ?  =  False && 3 >= 12
  ?  =  False
  ?  otherwise = True
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f m n p | m >= n && m >= p  =  m
        | n >= m && n >= p  =  n
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Lazy evaluation:

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f (2+3) (4-1) (3+9)
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  ?  =  True   && 5 >= 3+9
  ?  =  5 >= 3+9
  ?  =  5 >= 12
  ?  =  False
  ?  3 >= 5 && 3 >= 12
  ?  =  False && 3 >= 12
  ?  =  False
  ?  otherwise = True
=  12
```

`where`

Same principle: definitions in `where` clauses are only evaluated when needed and only as much as needed.

Lambda

Haskell never reduces inside a lambda

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Example: `\x -> False && x` cannot be reduced

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- Functions are black boxes
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Example:

```
(\x -> False && x) True  =  False && True  =  False
```

Built-in functions

Arithmetic operators and other built-in functions evaluate their arguments first

Example

`3 * 5` is a redex

Built-in functions

Arithmetic operators and other built-in functions evaluate their arguments first

Example

`3 * 5` is a redex

`0 * head (...)` is not a redex

Predefined functions from Prelude

They behave like their Haskell definition:

```
(&&) :: Bool -> Bool -> Bool
True  && y  =  y
False && y  =  False
```

Slogan

Lazy evaluation evaluates an expression only when needed and only as much as needed.

Slogan

Lazy evaluation evaluates an expression only when needed and only as much as needed.

(*"Call by need"*)

12.1 Applications of lazy evaluation

Minimum of a list

```
min  =  head . inSort
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inSort :: Ord a => [a] -> [a]
inSort []       =  []
inSort (x:xs)   =  ins x (inSort xs)
```


Minimum of a list

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min  =  head . inSort

inSort :: Ord a => [a] -> [a]
inSort []       =  []
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```


```
ins :: Ord a => a -> [a] -> [a]
ins x []   =  [x]
ins x (y:ys) | x <= y      =  x : y : ys
             | otherwise   =  y : ins x ys
```


Minimum of a list

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min  =  head . inSort

inSort :: Ord a => [a] -> [a]
inSort []       =  []
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ins x (y:ys) | x <= y      =  x : y : ys
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```


```
⟹  inSort [6,1,7,5]
    =  ins 6 (ins 1 (ins 7 (ins 5 [])))
```

```
min [6,1,7,5]   =   head(inSort [6,1,7,5])
```

```
min [6,1,7,5]  =  head(inSort [6,1,7,5])
=  head(ins 6 (ins 1 (ins 7 (ins 5 []))))
```

```
min [6,1,7,5]  =  head(inSort [6,1,7,5])
=  head(ins 6 (ins 1 (ins 7 (ins 5 []))))
=  head(ins 6 (ins 1 (ins 7 (5 : []))))
```

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min [6,1,7,5]  =  head(inSort [6,1,7,5])
=  head(ins 6 (ins 1 (ins 7 (ins 5 []))))
=  head(ins 6 (ins 1 (ins 7 (5 : []))))
```

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min [6,1,7,5]  =  head(inSort [6,1,7,5])
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=  head(ins 6 (ins 1 (5 : ins 7 [])))
```

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min [6,1,7,5]  =  head(inSort [6,1,7,5])
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ins x []  =  [x]
ins x (y:ys) | x <= y      =  x : y : ys
             | otherwise   =  y : ins x ys
```

```
min [6,1,7,5]  =  head(inSort [6,1,7,5])
=  head(ins 6 (ins 1 (ins 7 (ins 5 []))))
=  head(ins 6 (ins 1 (ins 7 (5 : []))))
=  head(ins 6 (ins 1 (5 : ins 7 [])))
=  head(ins 6 (1 : 5 : ins 7 []))
```

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min [6,1,7,5]  =  head(inSort [6,1,7,5])
=  head(ins 6 (ins 1 (ins 7 (ins 5 []))))
=  head(ins 6 (ins 1 (ins 7 (5 : []))))
=  head(ins 6 (ins 1 (5 : ins 7 [])))
=  head(ins 6 (1 : 5 : ins 7 []))
=  head(1 : ins 6 (5 : ins 7 []))
```

```
min [6,1,7,5]  =  head(inSort [6,1,7,5])
=  head(ins 6 (ins 1 (ins 7 (ins 5 []))))
=  head(ins 6 (ins 1 (ins 7 (5 : []))))
=  head(ins 6 (ins 1 (5 : ins 7 [])))
=  head(ins 6 (1 : 5 : ins 7 []))
=  head(1 : ins 6 (5 : ins 7 []))
=  1
```


```
min [6,1,7,5]  =  head(inSort [6,1,7,5])
=  head(ins 6 (ins 1 (ins 7 (ins 5 []))))
=  head(ins 6 (ins 1 (ins 7 (5 : []))))
=  head(ins 6 (ins 1 (5 : ins 7 [])))
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=  head(1 : ins 6 (5 : ins 7 []))
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```


Lazy evaluation needs only linear time


```
min [6,1,7,5]  =  head(inSort [6,1,7,5])
=  head(ins 6 (ins 1 (ins 7 (ins 5 []))))
=  head(ins 6 (ins 1 (ins 7 (5 : []))))
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=  1
```

Lazy evaluation needs only linear time

although `inSort` is quadratic
because the sorted list is never constructed completely


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min [6,1,7,5]  =  head(inSort [6,1,7,5])
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=  head(1 : ins 6 (5 : ins 7 []))
=  1
```

Lazy evaluation needs only linear time
although `inSort` is quadratic
because the sorted list is never constructed completely

Warning: this depends on the exact algorithm and does not work so nicely with all sorting functions!

Maximum of a list

```
max  =  last . inSort
```

Maximum of a list

```
max  =  last . inSort
```

Complexity?

Takeuchi Function

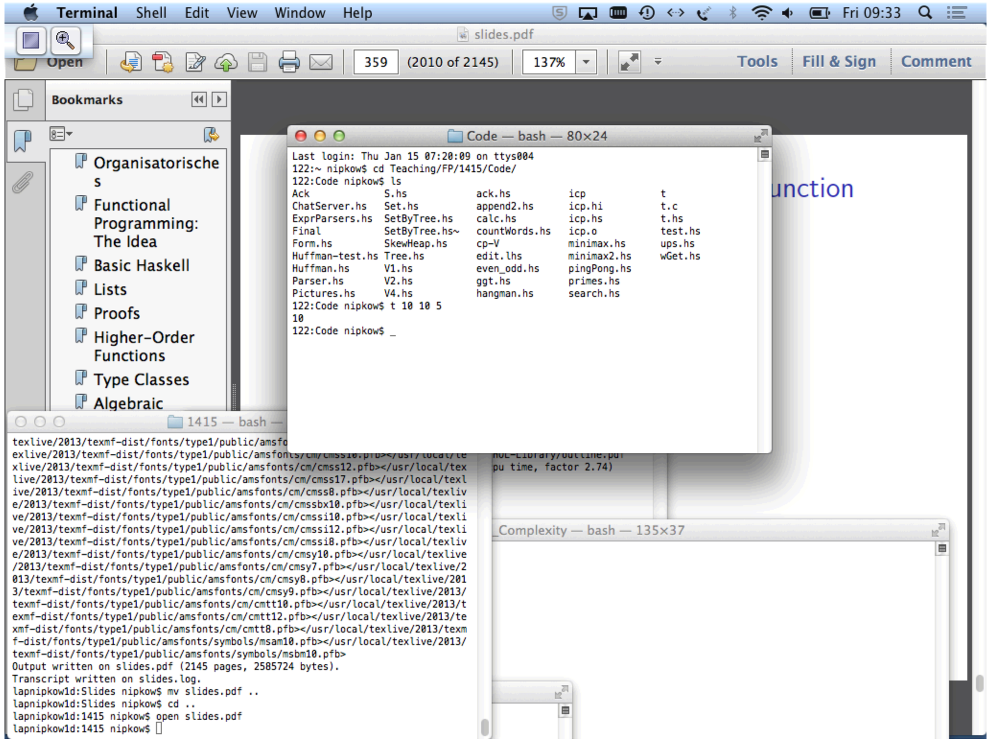
```
t :: Int -> Int -> Int -> Int
t x y z | x <= y     = y
        | otherwise = t (t (x-1) y z)
                        (t (y-1) z x)
                        (t (z-1) x y)
```

Takeuchi Function

```
t :: Int -> Int -> Int -> Int
t x y z | x <= y     = y
        | otherwise = t (t (x-1) y z)
                        (t (y-1) z x)
                        (t (z-1) x y)
```

In C:

```
int t(int x, int y, int z) {
  if (x <= y)
    return y;
  else
    return t(t(x-1, y, z), t(y-1, z, x), t(z-1, x, y));
}
```


Terminal Shell Edit View Window Help Fri 09:33
slides.pdf
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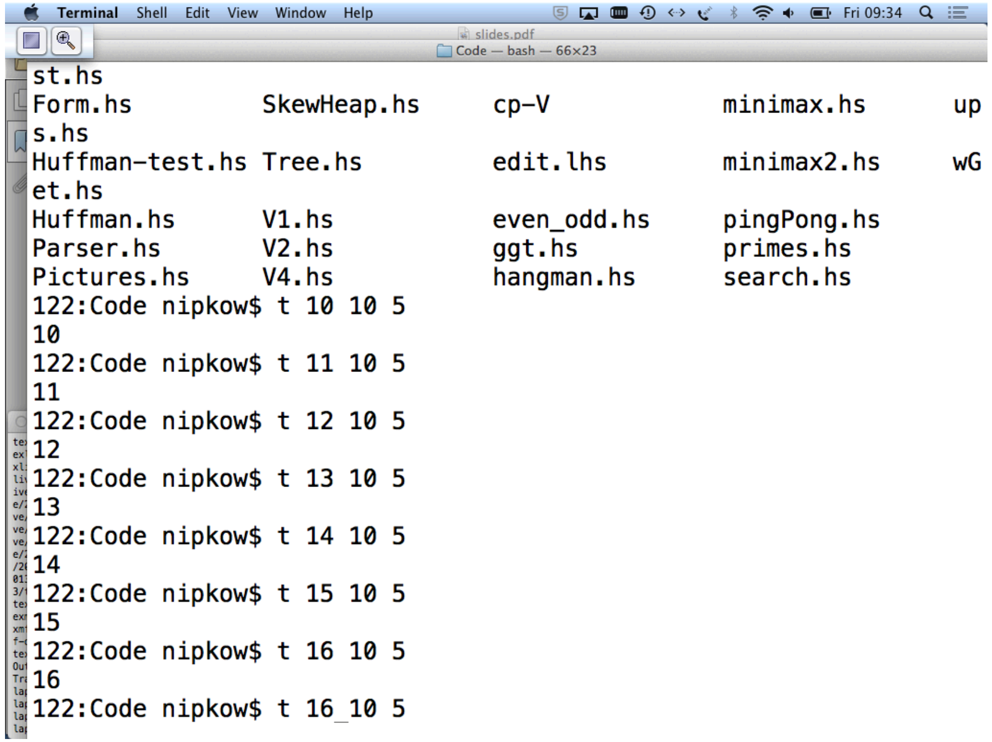
Organisatorisches
Functional Programming: The Idea
Basic Haskell
Lists
Proofs
Higher-Order Functions
Type Classes
Algebraic
Code — bash — 80×24
Complexity — bash — 135×37
1415 — bash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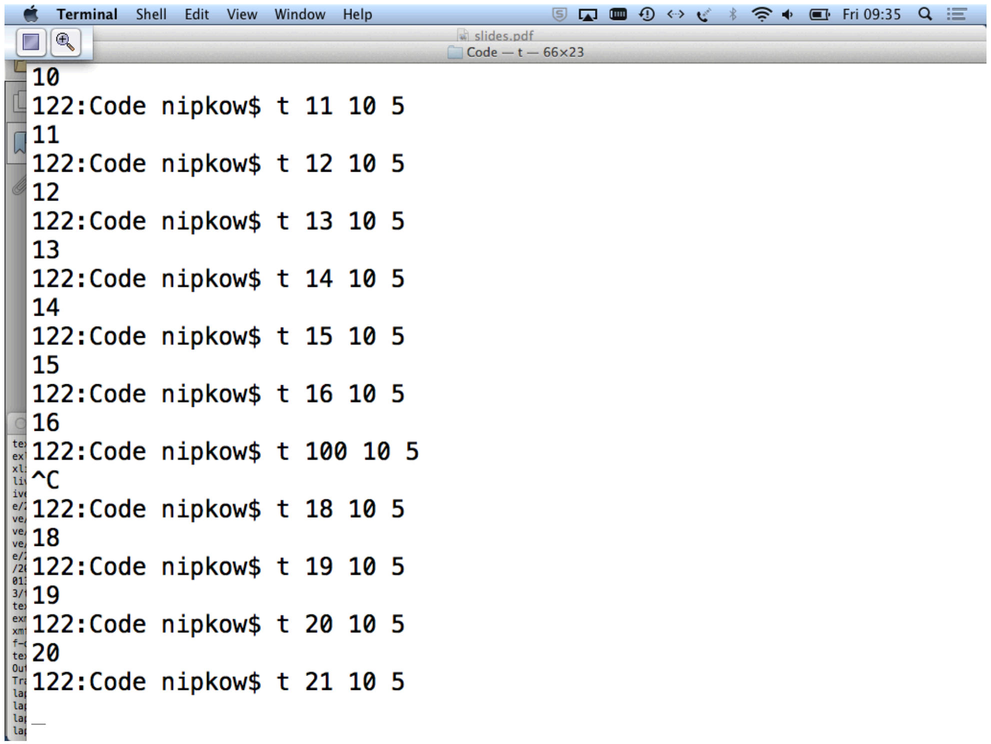
Code — bash — 66×23
st.hs
Form.hs SkewHeap.hs cp-V minimax.hs up
s.hs
Huffman-test.hs Tree.hs edit.lhs minimax2.hs wG
et.hs
Huffman.hs V1.hs even_odd.hs pingPong.hs
Parser.hs V2.hs ggt.hs primes.hs
Pictures.hs V4.hs hangman.hs search.hs
122:Code nipkow$ t 10 10 5
10
122:Code nipkow$ t 11 10 5
11
122:Code nipkow$ t 12 10 5
12
122:Code nipkow$ t 13 10 5
13
122:Code nipkow$ t 14 10 5
14
122:Code nipkow$ t 15 10 5
15
122:Code nipkow$ t 16 10 5
16
122:Code nipkow$ t 16_10 5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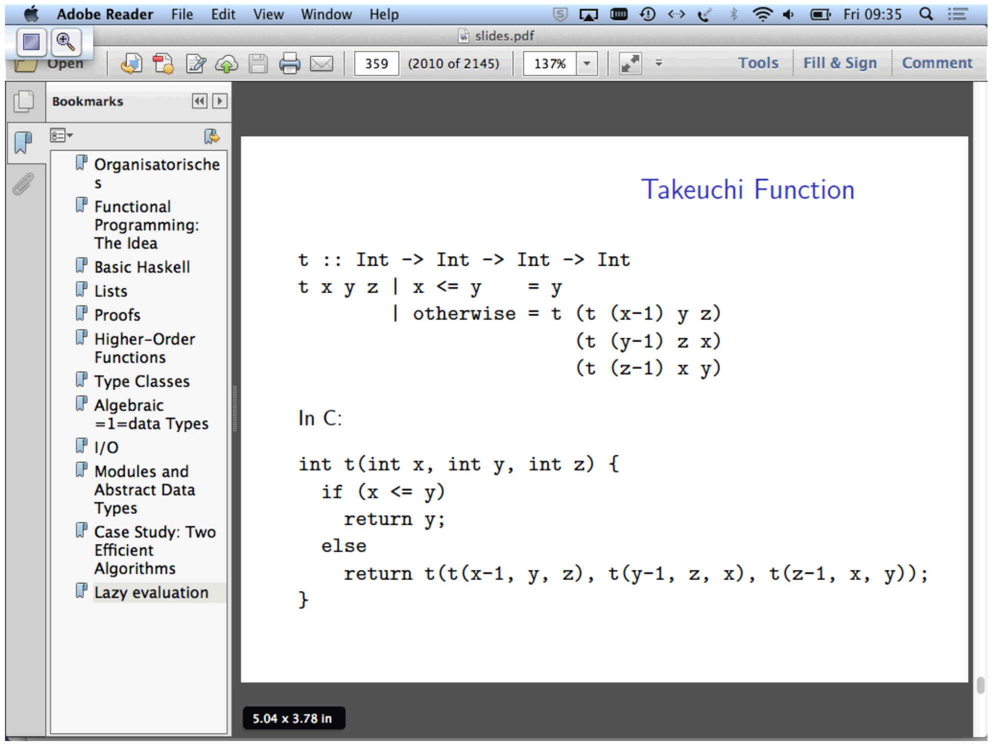
Code — t — 66×23
10
122:Code nipkow$ t 11 10 5
11
122:Code nipkow$ t 12 10 5
12
122:Code nipkow$ t 13 10 5
13
122:Code nipkow$ t 14 10 5
14
122:Code nipkow$ t 15 10 5
15
122:Code nipkow$ t 16 10 5
16
122:Code nipkow$ t 100 10 5
^C
122:Code nipkow$ t 18 10 5
18
122:Code nipkow$ t 19 10 5
19
122:Code nipkow$ t 20 10 5
20
122:Code nipkow$ t 21 10 5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Organisatorisches
Functional Programming: The Idea
Basic Haskell
Lists
Proofs
Higher-Order Functions
Type Classes
Algebraic =1=data Types
I/O
Modules and Abstract Data Types
Case Study: Two Efficient Algorithms
Lazy evaluation
Takeuchi Function
t :: Int -> Int -> Int -> Int
t x y z | x <= y = y
| otherwise = t (t (x-1) y z)
(t (y-1) z x)
(t (z-1) x y)
In C:
int t(int x, int y, int z) {
if (x <= y)
return y;
else
return t(t(x-1, y, z), t(y-1, z, x), t(z-1, x, y));
}
5.04 x 3.78 in

```
10
122:Code nipkow$ t 11 10 5
11
122:Code nipkow$ t 12 10 5
12
122:Code nipkow$ t 13 10 5
13
122:Code nipkow$ t 14 10 5
14
122:Code nipkow$ t 15 10 5
15
122:Code nipkow$ t 16 10 5
16
122:Code nipkow$ t 100 10 5
^C
122:Code nipkow$ t 18 10 5
18
122:Code nipkow$ t 19 10 5
19
122:Code nipkow$ t 20 10 5
20
122:Code nipkow$ t 21 10 5
_
```

```
18
122:Code nipkow$ t 19 10 5
19
122:Code nipkow$ t 20 10 5
20
122:Code nipkow$ t 21 10 5
^C
122:Code nipkow$ ghci
GHCi, version 7.6.3: http://www.haskell.org/ghc/  :? for help
Loading package ghc-prim ... linking ... done.
Loading package integer-gmp ... linking ... done.
Loading package base ... linking ... done.
Prelude> t 21 10 5

<interactive>:2:1: Not in scope: `t'
Prelude> :l t.hs
[1 of 1] Compiling Main             ( t.hs, interpreted )
Ok, modules loaded: Main.
*Main> t 21 10 5
21
*Main> t 100 10 5
100
*Main> _
```

Takeuchi Function

```
t :: Int -> Int -> Int -> Int
t x y z | x <= y     = y
        | otherwise = t (t (x-1) y z)
                        (t (y-1) z x)
                        (t (z-1) x y)
```

In C:

```
int t(int x, int y, int z) {
  if (x <= y)
    return y;
  else
    return t(t(x-1, y, z), t(y-1, z, x), t(z-1, x, y));
}
```

Try `t 15 10 0` — Haskell beats C!

12.2 Infinite lists

Example

A recursive definition

```
ones :: [Int]
ones  =  1 : ones
```

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that defines an infinite list of 1s:

```
ones  =  1 : ones  =  1 : 1 : ones  =  ...
```

What GHCi has to say about it:

```
> ones
[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1
```

Haskell lists can be finite or infinite

Example

A recursive definition

```
ones :: [Int]
ones  =  1 : ones
```

that defines an infinite list of 1s:

```
ones  =  1 : ones  =  1 : 1 : ones  =  ...
```

What GHCi has to say about it:

```
> ones
[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1
```

Haskell lists can be finite or infinite

Printing an infinite list does not terminate

But Haskell can compute with infinite lists, thanks to lazy evaluation:

```
> head ones
1
```

Remember:

Lazy evaluation evaluates an expression only as much as needed

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Remember:

Lazy evaluation evaluates an expression only as much as needed

Outermost reduction: `head ones = head (1 : ones) = 1`

But Haskell can compute with infinite lists, thanks to lazy evaluation:

```
> head ones
1
```

Remember:

Lazy evaluation evaluates an expression only as much as needed

Outermost reduction: `head ones = head (1 : ones) = 1`

Innermost reduction:

```
head ones
= head (1 : ones)
= head (1 : 1 : ones)
= ...
```

Haskell lists are never actually infinite but only potentially infinite

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Lazy evaluation computes as much of the infinite list as needed

This is how partially evaluated lists are represented internally:

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Lazy evaluation computes as much of the infinite list as needed

This is how partially evaluated lists are represented internally:

1 : 2 : 3 : | code pointer to compute rest |

Why (potentially) infinite lists?

- They come for free with lazy evaluation

Why (potentially) infinite lists?

- They come for free with lazy evaluation
- They increase modularity:
  list producer does not need to know
  how much of the list the consumer wants

Example: The sieve of Eratosthenes

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1. Create the list 2, 3, 4, ...
2. Output the first value p in the list as a prime.
3. Delete all multiples of p from the list

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3 5 7 9 11 ...

2

In Haskell:

```
primes :: [Int]
primes  =  sieve [2..]
```

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primes :: [Int]
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```

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sieve (p:xs)  =  p : sieve [x | x <- xs, x `mod` p /= 0]
```

Lazy evaluation:

```
primes  =  sieve [2..]  =  sieve (2:[3..])
```

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Lazy evaluation:

```
primes  =  sieve [2..]  =  sieve (2:[3..])
=  2 : sieve [x | x <- [3..], x `mod` 2 /= 0]
=  2 : sieve [x | x <- 3:[4..],  x `mod` 2 /= 0]
```

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=  2 : sieve [x | x <- 3:[4..],  x `mod` 2 /= 0]
=  2 : sieve (3 : [x | x <- [4..],  x `mod` 2 /= 0])
```


```
=  2 : 3 : sieve [x | x <- [x|x <- [4..],  x `mod` 2 /= 0]
                      x `mod` 3 /= 0]
```

In Haskell:

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primes :: [Int]
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=  2 : sieve (3 : [x | x <- [4..],  x `mod` 2 /= 0])
=  2 : 3 : sieve [x | x <- [x|x <- [4..],  x `mod` 2 /= 0]
                      x `mod` 3 /= 0]
```


```
=  ...
```

Modularity!

The first 10 primes:

```
> take 10 primes
[2,3,5,7,11,13,17,19,23,29]
```

The primes between 100 and 150:

Modularity!

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> takeWhile (<150) (dropWhile (<100) primes)
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```

All twin primes:

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> [(p,q) | (p,q) <-                         , p+2==q]
```

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> take 10 primes
[2,3,5,7,11,13,17,19,23,29]
```

The primes between 100 and 150:

```
> takeWhile (<150) (dropWhile (<100) primes)
[101,103,107,109,113,127,131,137,139,149]
```

All twin primes:

```
> [(p,q) | (p,q) <- zip primes (tail primes), p+2==q]
```

Sharing!

There is only one copy of `primes`

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Every time part of `primes` needs to be evaluated

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Example: when computing `take 5 primes`

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Every time part of `primes` needs to be evaluated

Example: when computing `take 5 primes`

primes is (invisibly!) updated to remember the evaluated part

Example: `primes = 2 : 3 : 5 : 7 : 11 : sieve ...`

Sharing!

There is only one copy of `primes`

Every time part of `primes` needs to be evaluated

Example: when computing `take 5 primes`

primes is (invisibly!) updated to remember the evaluated part

Example: `primes = 2 : 3 : 5 : 7 : 11 : sieve ...`

The next uses of `primes` are faster:

Example: now `primes !! 2` needs only 3 steps

Nothing special, just the automatic result of sharing

The list of Fibonacci numbers

Idea: 0 1 1 2 ...

The list of Fibonacci numbers

Idea:

```
    0 1 1 2 ...
+     0 1 1 ...
=   0 1 2 3 ...
```

From Prelude: `zipWith`

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    0 1 1 2 ...
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Example: `zipWith f [a1, a2, ...] [b1, b2, ...]`

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Example: `zipWith f [a1, a2, ...] [b1, b2, ...]`
`= [f a1 b1, f a2 b2, ...]`

```
fibs :: [Integer]
fibs = 0 :
```

The list of Fibonacci numbers

Idea:

```
    0 1 1 2 ...
+     0 1 1 ...
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Example: `zipWith f [a1, a2, ...] [b1, b2, ...]`
`= [f a1 b1, f a2 b2, ...]`

```
fibs :: [Integer]
fibs = 0 : 1 : zipWith (+) fibs
```

The list of Fibonacci numbers

Idea:

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    0 1 1 2 ...
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```

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Example: `zipWith f [a1, a2, ...] [b1, b2, ...]`
`= [f a1 b1, f a2 b2, ...]`

```
fibs :: [Integer]
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```

The list of Fibonacci numbers

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From Prelude: `zipWith`

Example: `zipWith f [a1, a2, ...] [b1, b2, ...]`
`= [f a1 b1, f a2 b2, ...]`

```
fibs :: [Integer]
fibs = 0 : 1 : zipWith (+) fibs (tail fibs)
```

How about

```
fibs = 0 : 1 : [x+y | x <- fibs, y <- tail fibs]
```