Script **generated by TTT**

Title: Seidl: Programmoptimierung (31.10.2012)

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Pages: 82

Bronislaw Knaster (1893-1980), topology

Theorem Knaster – Tarski

Assume $\mathbb{D}$ is a complete lattice. Then every monotonic function $f : \mathbb{D} \to \mathbb{D}$ has a least fixpoint $d_0 \in \mathbb{D}$.

Let $P = \{d \in \mathbb{D} \mid f\,d \sqsubseteq d\}$.

Then $d_0 = \bigsqcap P$.

$x \sqsupseteq f\,x$

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(1) $d_0 \in P$:

$f\, d_0 \sqsubseteq f\, d \sqsubseteq d$ for all $d \in P$

$\Longrightarrow$ $f\, d_0$ is a lower bound of P

$\Longrightarrow$ $f\, d_0 \sqsubseteq d_0$ since $d_0 = \bigsqcap P$

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(2) $f\, d_0 = d_0$:

$f\, d_0 \sqsubseteq d_0$ by (1)

$\Longrightarrow$ $f(f\, d_0) \sqsubseteq f\, d_0$ by monotonicity of f

$\Longrightarrow$ $f\, d_0 \in P$

$\Longrightarrow$ $d_0 \sqsubseteq f\, d_0$ and the claim follows :-)

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Assume $x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n$ $(*)$

is a system of constraints where all $f_i : \mathbb{D}^n \to \mathbb{D}$ are monotonic.

$\Longrightarrow$ least solution of $(*)$ $==$ least fixpoint of F :-)

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Theorem

- $\underline{\bot}, F\,\underline{\bot}, F^2\,\underline{\bot}, \ldots$ form an ascending chain :

$$\underline{\bot} \quad \sqsubseteq \quad F\,\underline{\bot} \quad \sqsubseteq \quad F^2\,\underline{\bot} \quad \sqsubseteq \quad \ldots$$

- If $F^k\,\underline{\bot} = F^{k+1}\,\underline{\bot}$, a solution is obtained which is the least one :-)
- If all ascending chains are finite, such a k always exists.

Proof

The first claim follows by complete induction:

Foundation: $F^0\,\underline{\bot} = \underline{\bot} \sqsubseteq F^1\,\underline{\bot}$:-)

Conclusion:

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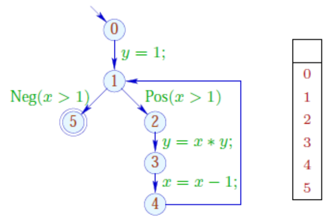

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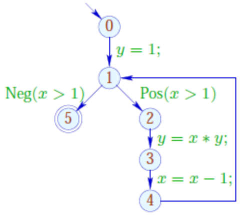

	1
0	$\emptyset$
1	$\{1, x > 1, x - 1\}$
2	$Expr$
3	$\{1, x > 1, x - 1\}$
4	$\{1\}$
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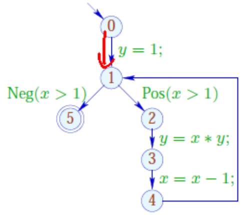

	1	2
0	$\emptyset$	$\emptyset$
1	$\{1, x > 1, x - 1\}$	$\{1\}$
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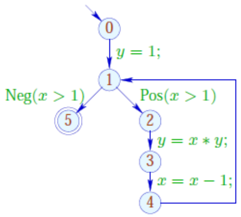

	1	2	3
0	$\emptyset$	$\emptyset$	$\emptyset$
1	$\{1, x > 1, x - 1\}$	$\{1\}$	$\{1\}$
2	$Expr$	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$
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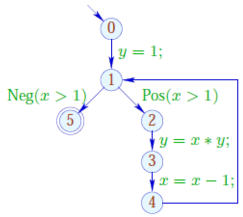

	1	2	3	4	5
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	
1	$\{1, x > 1, x - 1\}$	$\{1\}$	$\{1\}$	$\{1\}$	
2	$Expr$	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$	$\{1, x > 1\}$	
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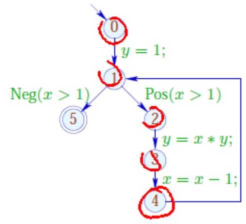

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Instead of accessing the values of the last iteration, always use the current values of unknowns :-)

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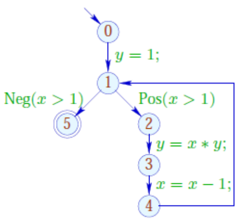

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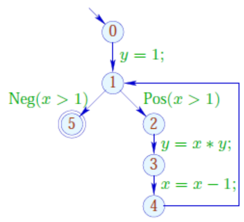

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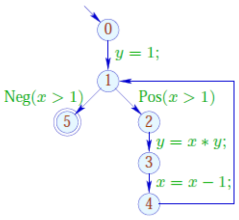

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The code for Round Robin Iteration in Java looks as follows:

```
for (i = 1; i ≤ n; i++) x_i = ⊥;
do {
    finished = true;
    for (i = 1; i ≤ n; i++) {
        new = f_i(x_1, ..., x_n);
        if (!(x_i ⊒ new)) {
            finished = false;
            x_i = x_i ⊔ new;
        }
    }
} while (!finished);
```


Correctness:

Assume $y_i^{(d)}$ is the i-th component of $F^d \perp$.

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The efficiency of RR-iteration depends on the ordering of the unknowns !!!

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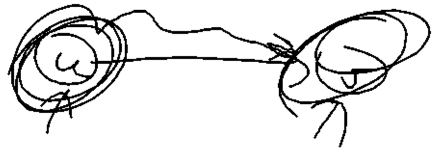
Good:

→ u before v, if $u \to^* v$;

→ entry condition before loop body :-)



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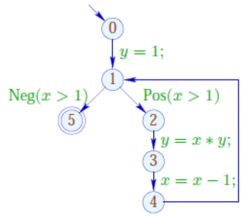

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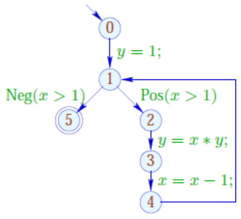
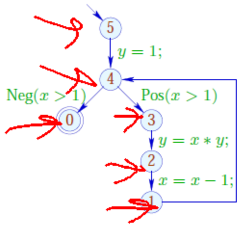
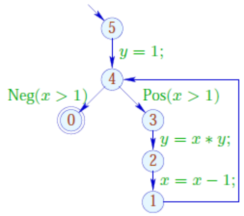


Bad:



Inefficient Round Robin Iteration:



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$\Longrightarrow$ significantly less efficient :-)

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For a complete lattice $\mathbb{D}$, consider systems:

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where $d_0 \in \mathbb{D}$ and all $[\![k]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ are monotonic ...

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$\Longrightarrow$ Monotonic Analysis Framework

Jeffrey D. Ullman, Stanford

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$$\mathcal{I}[v] \sqsupseteq \mathcal{I}^*[v] \qquad \text{for every } v$$

In particular: $\mathcal{I}[v] \sqsupseteq [\![\pi]\!]^\sharp\, d_0$ for every $\pi : start \to^* v$

Proof: Induction on the length of π.

Foundation: $\pi = \epsilon$ (empty path)

Then:

$$[\![\pi]\!]^\sharp\, d_0 = [\![\epsilon]\!]^\sharp\, d_0 = d_0 \sqsubseteq \mathcal{I}[start]$$

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Then:

$$[\![\pi']\!]^\sharp \, d_0 \sqsubseteq \mathcal{I}[u] \qquad \text{by I.H. for } \pi$$

$$\Longrightarrow \quad [\![\pi]\!]^\sharp \, d_0 = [\![k]\!]^\sharp \, ([\![\pi']\!]^\sharp \, d_0)$$

$$\sqsubseteq [\![k]\!]^\sharp \, (\mathcal{I}[u]) \qquad \text{since } [\![k]\!]^\sharp \text{ monotonic}$$

$$\sqsubseteq \mathcal{I}[v] \qquad \text{since } \mathcal{I} \text{ solution} \quad :-))$$

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Disappointment:

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With the notable exception when all functions $[\![k]\!]^\sharp$ are distributive ... :-)

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- distributive, if $f(\bigsqcup X) = \bigsqcup\{f\,x \mid x \in X\}$ for all $\emptyset \neq X \subseteq \mathbb{D}$;
- strict, if $f\,\bot = \bot$.
- totally distributive, if f is distributive and strict.

Remark:

If $f : \mathbb{D}_1 \to \mathbb{D}_2$ is distributive, then also monotonic :-)

Obviously: $a \sqsubseteq b$ iff $a \sqcup b = b$.

From that follows:

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  Distributivity:

$$\begin{aligned} f\,(x_1 \cup x_2) &= a \cap (x_1 \cup x_2) \cup b \\ &= a \cap x_1 \cup a \cap x_2 \cup b \\ &= f\,x_1 \cup f\,x_2 \end{aligned}$$

:-)

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  Distributivity:

$$f\,((1,4) \sqcup (4,1)) \quad = \quad f\,(4,4) \quad = \quad 8$$
$$\neq \quad 5 \quad = \quad f\,(1,4) \sqcup f\,(4,1) \qquad \text{:-)}$$

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Theorem

Kildall 1972

If all effects of edges $[\![k]\!]^\sharp$ are distributive, then: $\mathcal{I}^*[v] = \mathcal{I}[v]$ for all v .

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for all v .

Proof:

It suffices to prove that $\mathcal{I}^*$ is a solution :-)
For this, we show that $\mathcal{I}^*$ satisfies all constraints :-))

(1) We prove for *start* :

$$\begin{array}{lll} \mathcal{I}^*[start] & = & \bigsqcup\{[\![\pi]\!]^\sharp\, d_0 \mid \pi : start \to^* start\} \\ & \sqsupseteq & [\![\epsilon]\!]^\sharp\, d_0 \\ & \sqsupseteq & d_0 \quad :\text{-)} \end{array}$$

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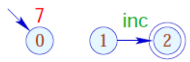
(2) For every $k = (u, \_, v)$ we prove:

$$\begin{array}{lll} \mathcal{I}^*[v] & = & \bigsqcup\{[\![\pi]\!]^\sharp\, d_0 \mid \pi : start \to^* v\} \\ & \sqsupseteq & \bigsqcup\{[\![\pi' k]\!]^\sharp\, d_0 \mid \pi' : start \to^* u\} \\ & = & \bigsqcup\{[\![k]\!]^\sharp\, ([\![\pi']\!]^\sharp\, d_0) \mid \pi' : start \to^* u\} \\ & = & [\![k]\!]^\sharp\, (\bigsqcup\{[\![\pi']\!]^\sharp\, d_0 \mid \pi' : start \to^* u\}) \\ & = & [\![k]\!]^\sharp\, (\mathcal{I}^*[u]) \end{array}$$

since $\{\pi' \mid \pi' : start \to^* u\}$ is non-empty :-)

Caveat:

- Reachability of all program points cannot be abandoned! Consider:



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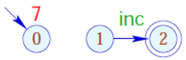

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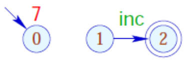

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- Unreachable program points can always be thrown away :-)

Summary and Application:

→ The effects of edges of the analysis of availability of expressions are distributive:

$$\begin{array}{lll} (a \cup (x_1 \cap x_2)) \backslash b & = & ((a \cup x_1) \cap (a \cup x_2)) \backslash b \\ & = & ((a \cup x_1) \backslash b) \cap ((a \cup x_2) \backslash b) \end{array}$$