Script **generated by TTT**

Title: Seidl: Programmoptimierung (12.11.2012)

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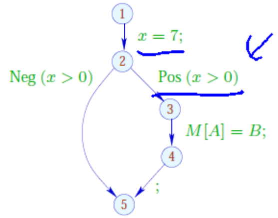
Pages: 63

At *start*, we have $D_\top = \{x \mapsto \top \mid x \in \mathit{Vars}\}$.

Example:



1	$\{x \mapsto \top\}$
2	$\{x \mapsto 7\}$
3	$\{x \mapsto 7\}$
4	$\{x \mapsto 7\}$
5	$\bot \sqcup \{x \mapsto 7\} = \{x \mapsto 7\}$

Patrick Cousot, ENS, Paris

The abstract effects of edges $[\![k]\!]^\sharp$ are again composed to the effects of paths $\pi = k_1 \ldots k_r$ by:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \ldots \circ [\![k_1]\!]^\sharp \quad : \mathbb{D} \to \mathbb{D}$$

Idea for Correctness: Abstract Interpretation

Cousot, Cousot 1977

Establish a description relation Δ between the concrete values and their descriptions with:

$$x \,\Delta\, a_1 \quad \wedge \quad a_1 \sqsubseteq a_2 \quad \Longrightarrow \quad x \,\Delta\, a_2$$

Concretization: $\gamma\, a = \{x \mid x \,\Delta\, a\}$

// returns the set of described values :-)

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(1) Values: $\Delta \subseteq \mathbb{Z} \times \mathbb{Z}^\top$

$z \,\Delta\, a$ iff $z = a \vee a = \top$

Concretization:

$$\gamma\, a = \begin{cases} \{a\} & \text{if} \quad a \sqsubset \top \\ \mathbb{Z} & \text{if} \quad a = \top \end{cases}$$

$x \in \mathbb{Z}$:

$x \,\Delta\, x$

$\wedge\; \forall x : x \,\Delta\, \top$

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$\gamma\, 5 = \{5\}$

$\gamma\, \top = \mathbb{Z}$

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(2) Variable Assignments: $\Delta \subseteq (\mathit{Vars} \to \mathbb{Z}) \times (\mathit{Vars} \to \mathbb{Z}^\top)_\perp$

$\rho \,\Delta\, D$ iff $D \neq \perp \wedge \rho\, x \sqsubseteq D\, x \quad (x \in \mathit{Vars})$

Concretization:

$$\gamma\, D = \begin{cases} \emptyset & \text{if} \quad D = \perp \\ \{\rho \mid \forall\, x : \; (\rho\, x) \,\Delta\, (D\, x)\} & \text{otherwise} \end{cases}$$

Example: $\{x \mapsto 1, y \mapsto -7\} \;\Delta\; \{x \mapsto \top, y \mapsto -7\}$

(3) States:

$$\Delta \;\subseteq\; ((\mathit{Vars} \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z})) \times (\mathit{Vars} \to \mathbb{Z}^{\top})_{\perp}$$

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We show:

(∗) If $s \;\Delta\; D$ and $[\![\pi]\!]\, s$ is defined, then:

$$([\![\pi]\!]\, s) \;\Delta\; ([\![\pi]\!]^{\sharp}\, D)$$

$$\begin{tikzcd} s \arrow[r, "{[\![\pi]\!]}"] \arrow[d, dash, "\Delta"'] & s_1 \arrow[d, dash, dotted, "\Delta"] \\ D \arrow[r, "{[\![\pi]\!]^{\sharp}}"'] & D_1 \end{tikzcd}$$

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In practice, this means, e.g., that $D\, x = -7$ implies:

$$\begin{array}{rcll} \rho'\, x & = & -7 & \text{for all} \quad \rho' \in \gamma\, D \\ \Longrightarrow \quad \rho_1\, x & = & -7 & \text{for} \quad (\rho_1, \_) = [\![\pi]\!]\, s \end{array}$$

To prove $(*)$, we show for every edge k :

$$\begin{tikzcd} & s \arrow[r, "{[\![k]\!]}"] \arrow[d, dash, "\Delta"'] & s_1 \arrow[d, dash, dotted, "\Delta"] \\ (**) & D \arrow[r, "{[\![k]\!]^\sharp}"'] & D_1 \end{tikzcd}$$

Then $(*)$ follows by induction :-)

To prove $(**)$, we show for every expression e :

$(***)$ $\quad ([\![e]\!]\,\rho)\ \Delta\ ([\![e]\!]^\sharp\,D)\quad$ whenever $\quad \rho\ \Delta\ D$

Base ✓

$= \cap$

$= D$

$\pi' = \pi k$

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This precisely was how we have defined the operators $\Box^\sharp$:-)

Now, $(**)$ is proved by case distinction on the edge labels *lab* .

Let $s = (\rho, \mu) \;\Delta\; D$. In particular, $\bot \neq D \;:\; Vars \to \mathbb{Z}^\top$

Case $\boxed{x = e;}$:

$$\begin{array}{lll} \rho_1 & = & \rho \oplus \{x \mapsto [\![e]\!]\,\rho\} \qquad \mu_1 \;=\; \mu \\ D_1 & = & D \oplus \{x \mapsto [\![e]\!]^\sharp\,D\} \end{array}$$

$\Longrightarrow \qquad (\rho_1, \mu_1) \;\Delta\; D_1$ $\qquad$ No $\rho : \rho \;\Delta\; \bot$

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Case $\boxed{x = M[e];}$:

$$\begin{array}{lll} \rho_1 & = & \rho \oplus \{x \mapsto \mu\,([\![e]\!]^\sharp \rho)\} \qquad\qquad \mu_1 \;=\; \mu \\ D_1 & = & D \oplus \{x \mapsto \top\} \end{array}$$

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Case $\boxed{M[e_1] = e_2;}$:

$$\begin{array}{lll} \rho_1 & = & \rho \qquad\qquad \mu_1 \;=\; \mu \oplus \{[\![e_1]\!]^\sharp \rho \mapsto [\![e_2]\!]^\sharp \rho\} \\ D_1 & = & D \end{array}$$

$$\Longrightarrow \qquad (\rho_1, \mu_1) \;\Delta\; D_1$$

Case $\boxed{\mathrm{Neg}(e)}$: $\qquad (\rho_1, \mu_1) = s$ where:

$$\begin{array}{rlll} & 0 & = & [\![e]\!]\,\rho \\ & & \Delta & [\![e]\!]^\sharp\, D \\ \Longrightarrow & 0 & \sqsubseteq & [\![e]\!]^\sharp\, D \\ \Longrightarrow & \bot & \neq & D_1 \;=\; D \\ \Longrightarrow & \multicolumn{3}{l}{(\rho_1, \mu_1) \;\Delta\; D_1} \end{array}$$

Case $\boxed{\text{Pos}(e)}$: $(\rho_1, \mu_1) = s$ where:

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:-)

$(\ast\ast)$

We conclude: The assertion $(*)$ is true :-))

The MOP-Solution:

$$\mathcal{D}^*[v] \;=\; \bigsqcup \{ [\![\pi]\!]^\sharp\, D_\top \mid \pi : start \to^* v \}$$

where $D_\top\, x = \top \qquad (x \in Vars)$.

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$$([\![\pi]\!]\, s) \;\Delta\; (\mathcal{D}^*[v]) \sqsubseteq D[v]$$

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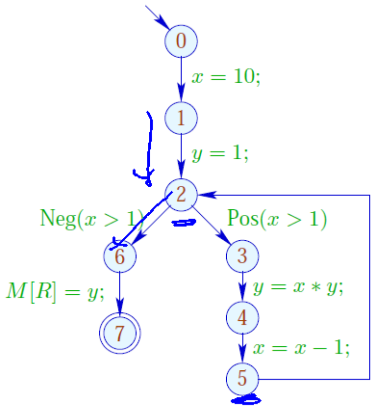
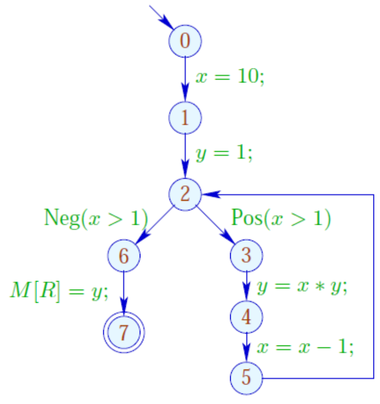
Example:



	1	
	x	y
0	$\top$	$\top$
1	10	$\top$
2	10	1
3	10	1
4	10	10
5	9	10
6	$\bot$	
7	$\bot$	

1: $x = 10$
$y = 1$

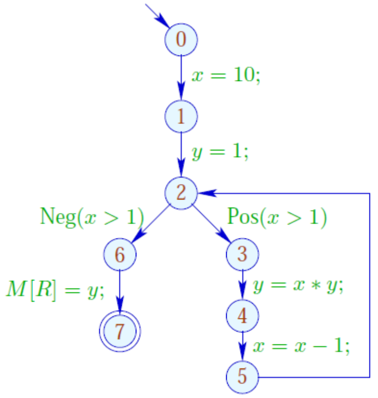
Example:



	1		2	
	x	y	x	y
0	$\top$	$\top$	$\top$	$\top$
1	10	$\top$	10	$\top$
2	10	1	$\top$	$\top$
3	10	1	$\top$	$\top$
4	10	10	$\top$	$\top$
5	9	10	$\top$	$\top$
6	$\bot$		$\top$	$\top$
7	$\bot$		$\top$	$\top$

5: $x = 9$
$y = 10$

Example:



	1		2		3	
	x	y	x	y	x	y
0	$\top$	$\top$	$\top$	$\top$		
1	10	$\top$	10	$\top$		
2	10	1	$\top$	$\top$		
3	10	1	$\top$	$\top$		
4	10	10	$\top$	$\top$	dito	
5	9	10	$\top$	$\top$		
6	$\bot$		$\top$	$\top$		
7	$\bot$		$\top$	$\top$		

Conclusion:

Although we compute with concrete values, we fail to compute everything :-(

The fixpoint iteration, at least, is guaranteed to terminate:

For n program points and m variables, we maximally need: $n \cdot (m + 1)$ rounds :-)

Caveat:

The effects of edge are not distributive !!!

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The effects of edge are not distributive !!!

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Counter Example: $\quad f = [\![x = x + y;]\!]^\sharp$

$$
\begin{aligned}
\text{Let} \quad D_1 &= \{x \mapsto 2, y \mapsto 3\} \\
D_2 &= \{x \mapsto 3, y \mapsto 2\} \\
\text{Dann} \quad f\,D_1 \sqcup f\,D_2 &= \{x \mapsto 5, y \mapsto 3\} \sqcup \{x \mapsto 5, y \mapsto 2\} \\
&= \{x \mapsto 5, y \mapsto \top\} \\
&\neq \{x \mapsto \top, y \mapsto \top\} \\
&= f\,\{x \mapsto \top, y \mapsto \top\} \\
&= f\,(D_1 \sqcup D_2)
\end{aligned}
$$

MoP↓

⊔!

const

:-((

We conclude:

The least solution $\mathcal{D}$ of the constraint system in general yields only an upper approximation of the MOP, i.e.,

$$\mathcal{D}^*[v] \quad \sqsubseteq \quad \mathcal{D}[v]$$

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$$\sqcap \Delta \quad \mathcal{D}^*[v] \quad \sqsubseteq \quad \mathcal{D}[v]$$

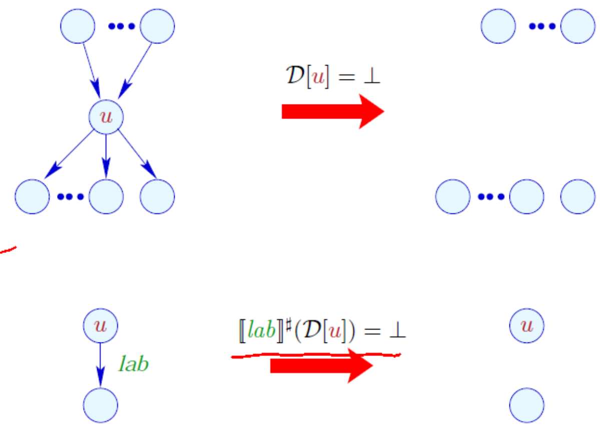
$\sqcap D$

As an upper approximation, $\mathcal{D}[v]$ nonetheless describes the result of every program execution π which reaches v :

$$([\![\pi]\!]\,(\rho, \mu)) \;\; \Delta \;\; (\mathcal{D}[v])$$

whenever $[\![\pi]\!]\,(\rho, \mu)$ is defined ;-))

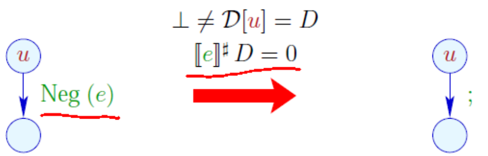
Transformation 4: Removal of Dead Code

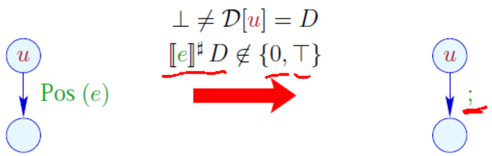


$u \neq \bot$

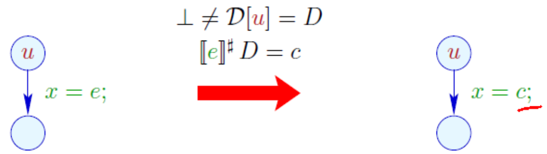
$v \neq \bot$

Transformation 4 (cont.): Removal of Dead Code





Transformation 4 (cont.): Simplified Expressions



Extensions:

- Instead of complete right-hand sides, also subexpressions could be simplified:

$$x + (3 * y) \xRightarrow{\{x \mapsto \top, y \mapsto 5\}} x + 15$$

... and further simplifications be applied, e.g.:

$$\begin{array}{lcl} x * 0 & \Longrightarrow & 0 \\ x * 1 & \Longrightarrow & x \\ x + 0 & \Longrightarrow & x \\ x - 0 & \Longrightarrow & x \\ & \ldots & \end{array}$$

- So far, the information of conditions has not yet be optimally exploited:

$$\begin{array}{l} \texttt{if } (x == 7) \\ \quad y = x + 3; \end{array}$$

Even if the value of x before the if statement is unknown, we at least know that x definitely has the value 7 — whenever the then-part is entered :-)

Therefore, we can define:

$$[\![\mathsf{Pos}\,(x == e)]\!]^\sharp\, D = \begin{cases} D & \text{if } [\![x == e]\!]^\sharp\, D = 1 \\ \bot & \text{if } [\![x == e]\!]^\sharp\, D = 0 \\ D_1 & \text{otherwise} \end{cases}$$

where

$$D_1 = D \oplus \{x \mapsto (D\,x \sqcap [\![e]\!]^\sharp\, D)\}$$

Extensions:

- Instead of complete right-hand sides, also subexpressions could be simplified:

$D: \{x \mapsto \top\}$

$\to$ Pos$(x == 7)$

$$x + (3 * y) \xRightarrow{\{x \mapsto \top, y \mapsto 5\}} x + 15$$

... and further simplifications be applied, e.g.:

$[\![x == 7]\!]^\sharp = \top$

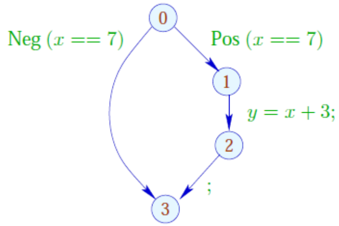
$= \top ==^\sharp 7$

$$\begin{array}{lcl} x * 0 & \Longrightarrow & 0 \\ x * 1 & \Longrightarrow & x \\ x + 0 & \Longrightarrow & x \\ x - 0 & \Longrightarrow & x \\ & \ldots & \end{array}$$

$x \mapsto \top \sqcap 7$
$= 7$

The effect of an edge labeled Neg $(x \neq e)$ is analogous :-)

Our Example:



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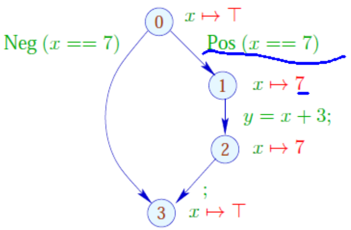

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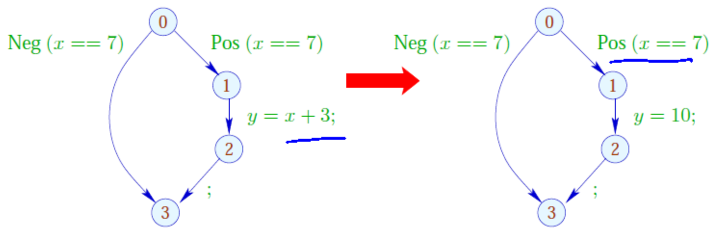

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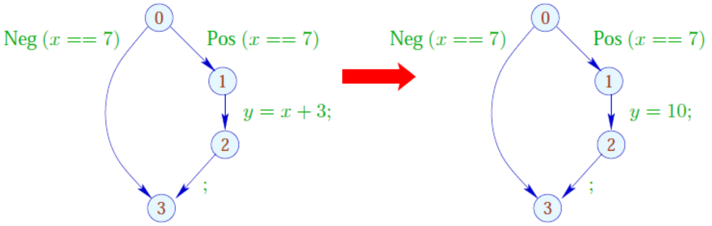

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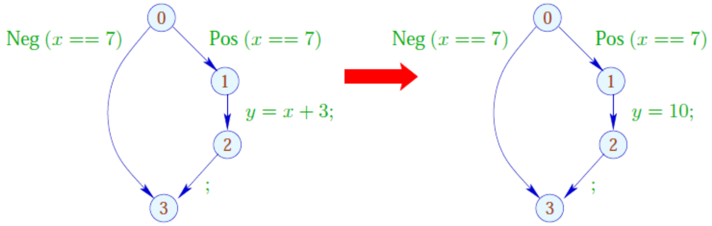

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1.5 Interval Analysis

Observation:

- Programmers often use global constants for switching debugging code on/off.

  $\Longrightarrow$

  Constant propagation is useful :-)

- In general, precise values of variables will be unknown — perhaps, however, a tight interval !!!

Example:

$$
\begin{array}{l}
\text{for } (i = 0; i < 42; i{+}{+}) \\
\quad \text{if } (0 \leq i \wedge i < 42)\{ \\
\qquad A_1 = A + i; \\
\qquad M[A_1] = i; \\
\quad \} \\
// \quad A \text{ start address of an array} \\
// \quad \text{if the array-bound check}
\end{array}
$$

Obviously, the inner check is superfluous :-)

Idea 1:

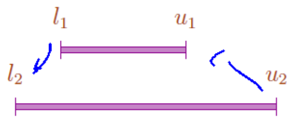
Determine for every variable x an (as tight as possible :-) interval of possible values:

$$\mathbb{I} = \{[l,u] \mid l \in \mathbb{Z} \cup \{-\infty\}, u \in \mathbb{Z} \cup \{+\infty\}, l \leq u\}$$

Partial Ordering:

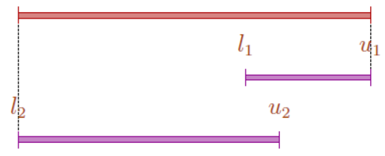
[8, 2]

$$[l_1,u_1] \sqsubseteq [l_2,u_2] \quad \text{iff} \quad l_2 \leq l_1 \wedge u_1 \leq u_2$$



Thus:

$$[l_1,u_1] \sqcup [l_2,u_2] = [l_1 \sqcap l_2, u_1 \sqcup u_2]$$



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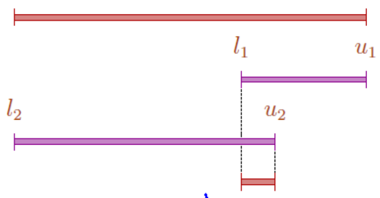
$$[l_1,u_1] \sqcap [l_2,u_2] = [l_1 \sqcup l_2, u_1 \sqcap u_2] \quad \text{whenever } (l_1 \sqcup l_2) \leq (u_1 \sqcap u_2)$$



Caveat:

→ $\mathbb{I}$ is not a complete lattice :-)

→ $\mathbb{I}$ has infinite ascending chains, e.g.,

$$[0,0] \sqsubset [0,1] \sqsubset [-1,1] \sqsubset [-1,2] \sqsubset \ldots$$

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while (true)
x = x+1;

[0,0] [0,1] [0,2]

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Description Relation:

$$z \,\Delta\, [l,u] \qquad \text{iff} \qquad l \le z \le u$$

Concretization:

$$\gamma\,[l,u] = \{z \in \mathbb{Z} \mid l \le z \le u\}$$

Example:

$$\begin{aligned} \gamma\,[0,7] &= \{0,\ldots,7\} \\ \gamma\,[0,\infty] &= \{0,1,2,\ldots,\} \end{aligned}$$

Computing with intervals:

Interval Arithmetic :-)

Addition:

$$\begin{aligned} [l_1,u_1] +^\sharp [l_2,u_2] &= [l_1+l_2, u_1+u_2] \qquad \text{where} \\ -\infty + \_ &= -\infty \\ +\infty + \_ &= +\infty \\ // \quad & -\infty + \infty \quad \text{cannot occur} \end{aligned}$$

:-)

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