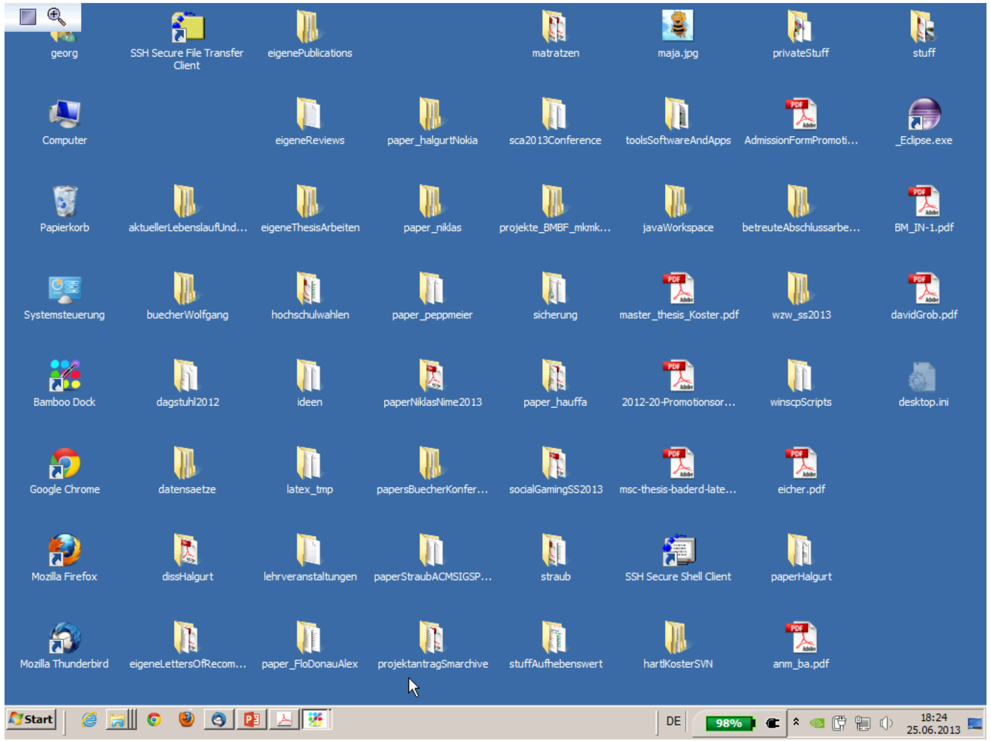
Script **generated by TTT**

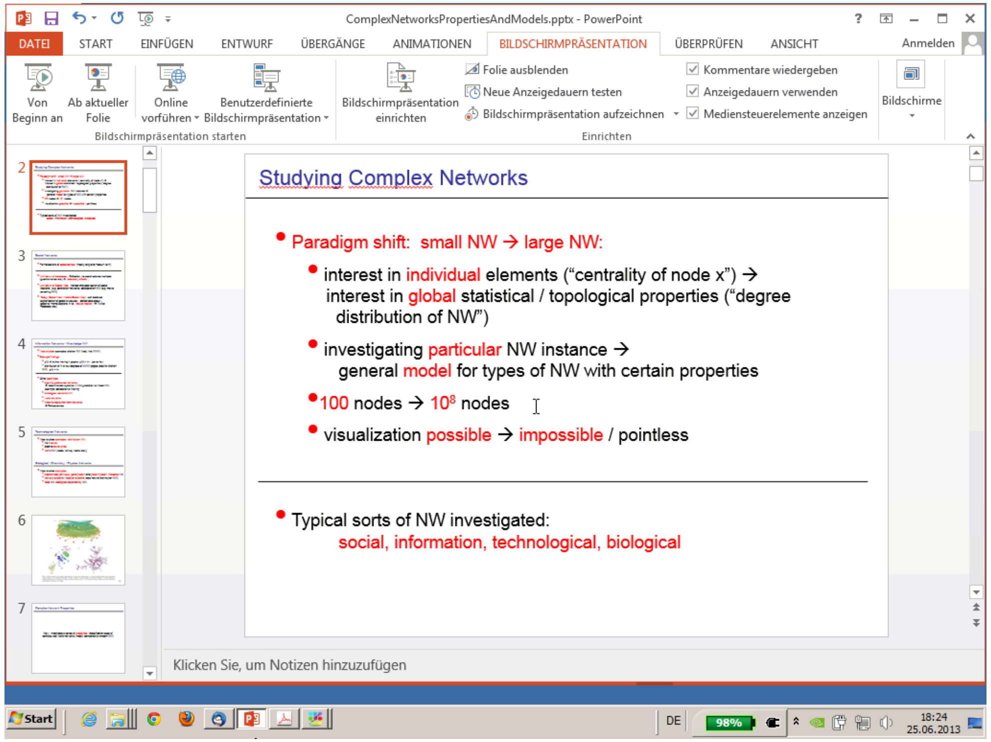


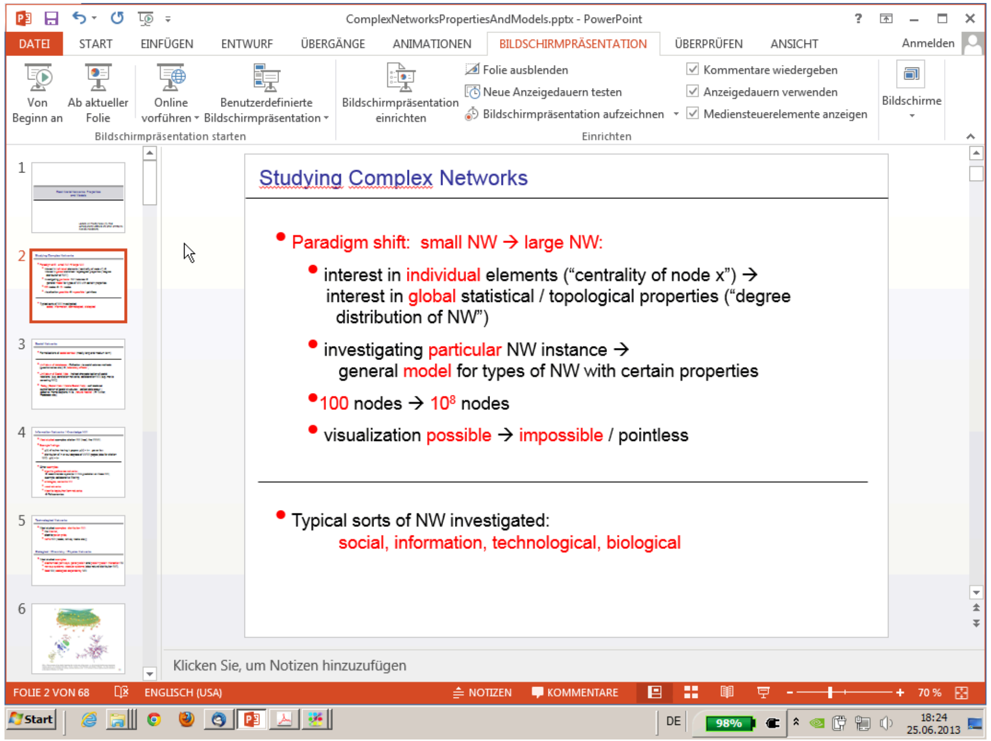
Title: profile1 (25.06.2013)

Date: Tue Jun 25 18:24:23 CEST 2013

Duration: 89:29 min

Pages: 100





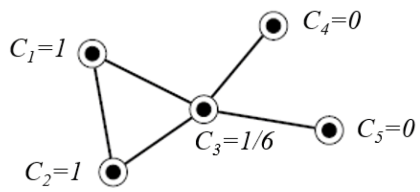
Real World Networks: Properties and Models

Lecture will mostly follow [1], thus corresponding citations are often omitted to increase readability

Transitivity / Clustering Coefficient

Example:

$$C^{(1)} = \frac{3\times \text{ number of triangles in the network}}{\text{number of connected triples of vertices}} = \frac{3\times 1}{8} = 0.375$$



$$C^{(2)} = \frac{1}{n}\sum_i C_i \quad \text{with} \quad C_i = \frac{\text{number of triangles connected to vertex } i}{\text{number of triples centered on vertex } i}$$

$C^{(2)} = 1/5\ (1 + 1 + 1/6 + 0 + 0) = 13/30 = 0.433333$

Mean Average Path Length

- "Small World Effect": $l(n)$ "small" → $l(n) \in O(\log(n))$
- undirected graph:

$$\ell = \frac{1}{\frac{1}{2}n(n+1)}\sum_{i\geq j} d_{ij}$$

formula also counts 0 distances from i to i: ½ n(n+1) = ½ n(n-1) + n

- Expression allowing for disconnected components (where $d_{ij}=\infty$ can occur): harmonic mean:

$$\ell^{-1} = \frac{1}{\frac{1}{2}n(n+1)}\sum_{i\geq j} d_{ij}^{-1}$$

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Transitivity / Clustering Coefficient

- Clustering coefficient (whole graph):

$$C = C^{(1)} = \frac{3\times \text{ number of triangles in the network}}{\text{number of connected triples of vertices}} = \frac{6\times \text{ number of triangles in the network}}{\text{number of paths of length two}}$$

p(FOAF)

- Clustering coefficient (Watts-Strogatz-version, for node i):

$$C_i = \frac{\text{number of triangles connected to vertex } i}{\text{number of triples centered on vertex } i} = \frac{|\{e_{\{kj\}} \mid v_k, v_j \in N_i\}|}{\frac{k_i(k_i-1)}{2}}$$

(see Introduction , k_i = degree of node i)

Clustering coefficient (Watts-Strogatz-version, for whole graph):

$$C = C^{(2)} = \frac{1}{n}\sum_i C_i$$

mean of ratio instead of ratio of means

Degree Distribution

- Notation:
  p(k) = p_k = fraction of nodes having degree k
- Cumulative distribution:

$$P_k = \sum_{k'=k}^{\infty} p_{k'}$$

- power law:

$$p_k \sim k^{-\alpha}$$

$$\rightarrow \quad P_k \sim \sum_{k'=k}^{\infty} k'^{-\alpha} \sim k^{-(\alpha-1)}$$

- exponential:

$$p_k \sim e^{-k/\kappa}$$

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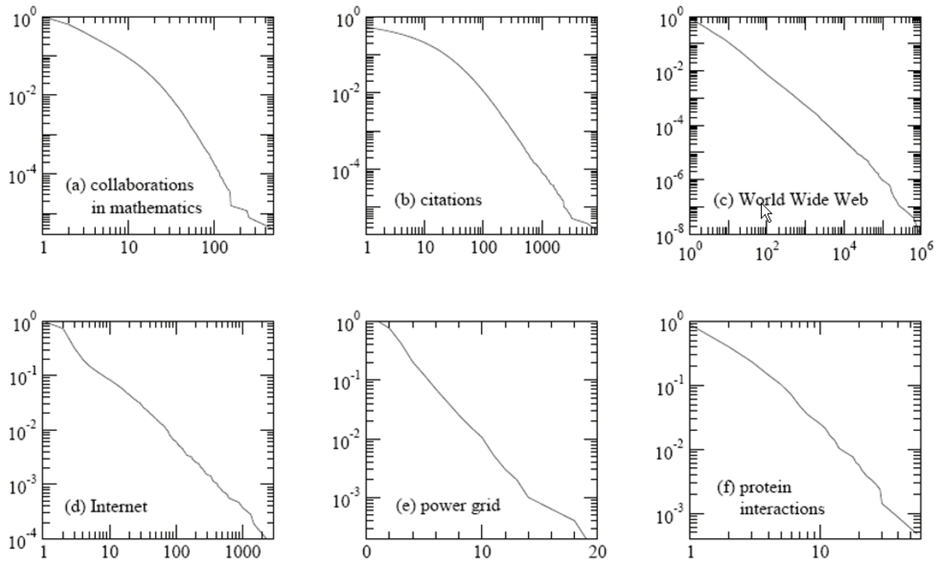
Degree Distribution

Cumulative distributions P_k of example real world NW



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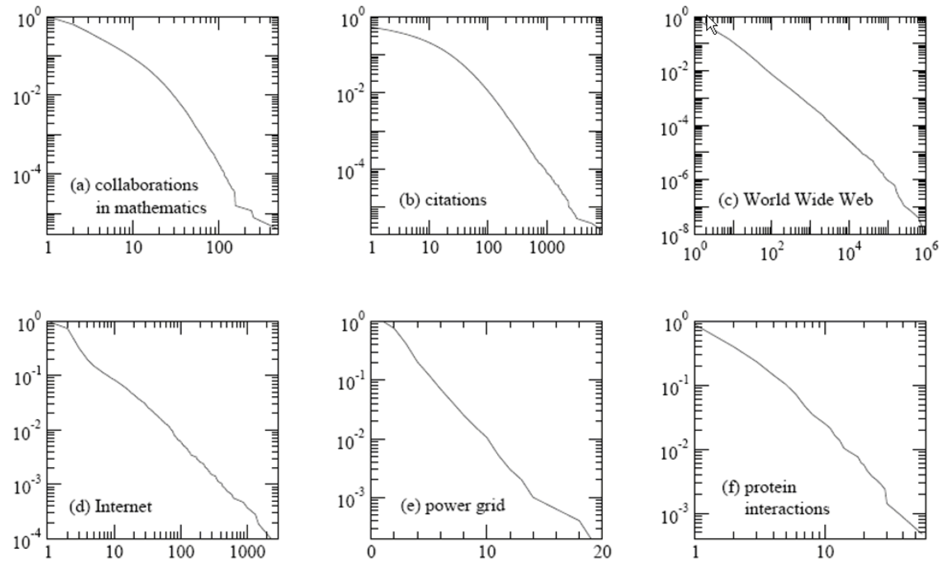

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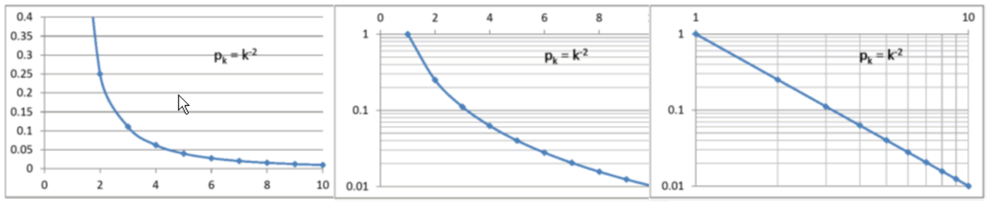

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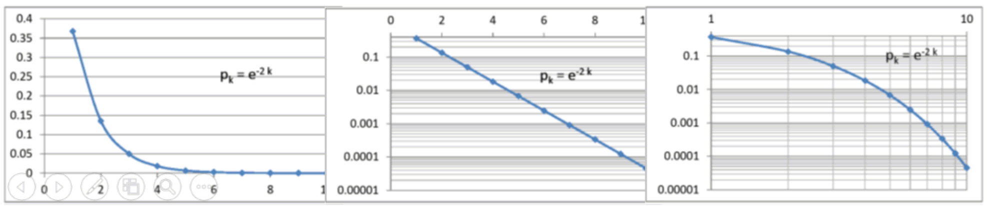

Degree Distribution

Examples:

- Power law: citation NW, WWW, Internet, metabolic NW, telephone call NW, human sexual contact NW etc.
- Exponential: power grid, railway NW
- Power law with exp. cut-offs: Movie co-actor NW

Maximum Degree

- „less or equal than one vertex with k_{max}“
→ $np_{k\_max} = 1$ → for power law $p_k = k^{-\alpha}$: $k_{max} \sim n^{1/\alpha}$
but: not very accurate estimation

- Other estimation:
  - prob p of „exactly m nodes with k and rest of nodes smaller than k“:
  $$\binom{n}{m} p_k^m (1 - P_k)^{n-m}$$
  - → prob of k being the highest degree in graph:
  $$h_k = \sum_{m=1}^{n} \binom{n}{m} p_k^m (1 - P_k)^{n-m} = (p_k + 1 - P_k)^n - (1 - P_k)^n$$
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Maximum Degree

- since h_k is small for small k and also for large k → take as k_{max} the modal value of h_k →

modal value : $\frac{d}{dk} h_k = 0$

Using $\mathrm{d}P_k/\mathrm{d}k = p_k$ we get

$$\frac{d}{dk} h_k = n \left[\left(\frac{\mathrm{d}p_k}{\mathrm{d}k} - p_k\right)(p_k + 1 - P_k)^{n-1} + p_k(1 - P_k)^{n-1}\right] = 0$$

or $k_{\max}$ is a solution of

$$\frac{\mathrm{d}p_k}{\mathrm{d}k} \simeq -np_k^2$$

(assuming: p_k is small for $k > k_{max}$ and that $np_k \ll 1$ and that $P_k \ll 1$)

→we get for power law $p_k \sim k^{-\alpha}$ that $k_{\max} \sim n^{1/(\alpha-1)}$

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Network Resilience

- What happens if nodes are removed? (interesting e.g. for vaccination effects in disease spreading in human contact networks)

- For power law networks:

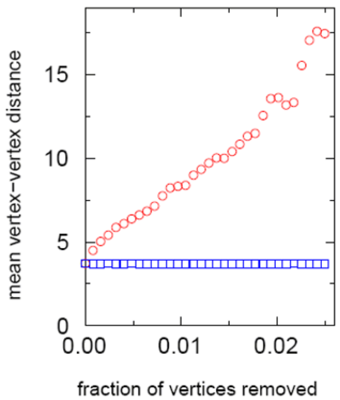
  remove random nodes :
  no effect on mean distances

  remove high degree nodes:
  drastic effect

- Interpretations:

  Internet is easy to attack

  Internet is not easy to attack



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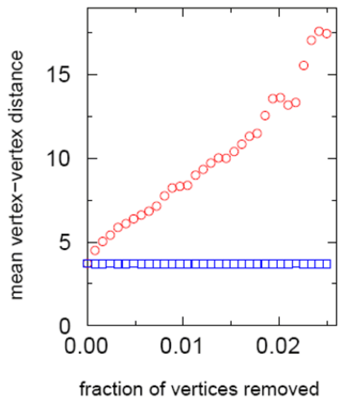

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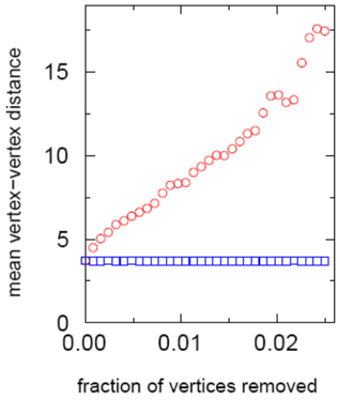

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Mixing Patterns

- Ecological NW, Internet, some social NW:
  Assortative Mixing (Homophily): Nodes attach to similar nodes / nodes of same class OR
  Disassortative Mixing (Heterophily): Nodes attach to nodes of different classes (almost n-partite behavior)

- Diassortativity:
  Food Web: Plants ←→ Herbivores ←→ Carnivores
  but few Plants ←→ Plants etc.
  Internet: Backbones provider ←→ ISP ←→ end user
  but few ISP ←→ ISP etc.

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$\mathbf{E} =$

		women			
		black	hispanic	white	other
men	black	506	32	69	26
	hispanic	23	308	114	38
	white	26	46	599	68
	other	10	14	47	32

TABLE III Couples in the study of Catania *et al.* [85] tabulated by race of either partner. After Morris [302]. [1]

- measure mixing: analogous to modularity: mixing matrix $\mathbf{e} = \frac{\mathbf{E}}{\|\mathbf{E}\|}$

→ $P(j|i) = e_{ij} / \sum_{j'} e_{ij'}$ $\quad \sum_{ij} e_{ij} = 1, \quad \sum_j P(j|i) = 1$

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TABLE III Couples in the study of Catania *et al.* [85] tabulated by race of either partner. After Morris [302].

- → first measure for Assortativity:

$$Q = \frac{\sum_i P(i|i) - 1}{N - 1}$$

issues: Asymmetry of E → two values;
Not respecting size of classes

- → second measure for Assortativity: (cmp. Modularity)

$$r = \frac{\mathrm{Tr}\,\mathbf{e} - \|\mathbf{e}^2\|}{1 - \|\mathbf{e}^2\|}$$

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Mixing Patterns

E =

		women			
		black	hispanic	white	other
men	black	506	32	69	26
	hispanic	23	308	114	38
	white	26	46	599	68
	other	10	14	47	32

TABLE III Couples in the study of Catania *et al.* [85] tabulated by race of either partner. After Morris [302].

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$$Q = \frac{\sum_i P(i|i) - 1}{N-1}$$

issues: Asymmetry of E → two values;
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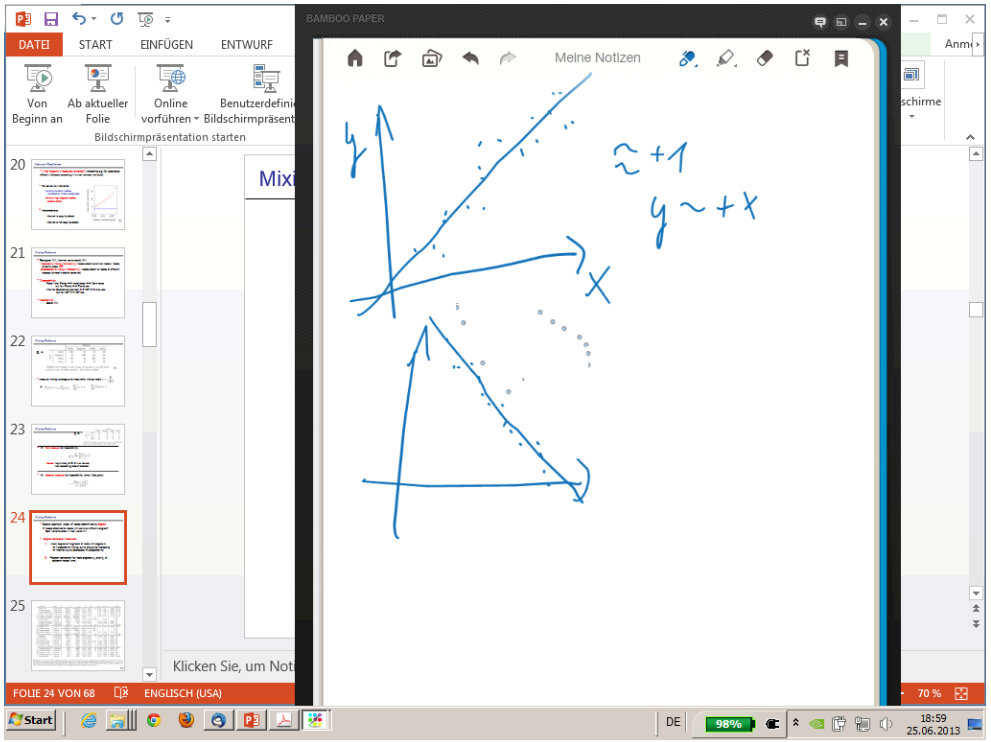
Mixing Patterns

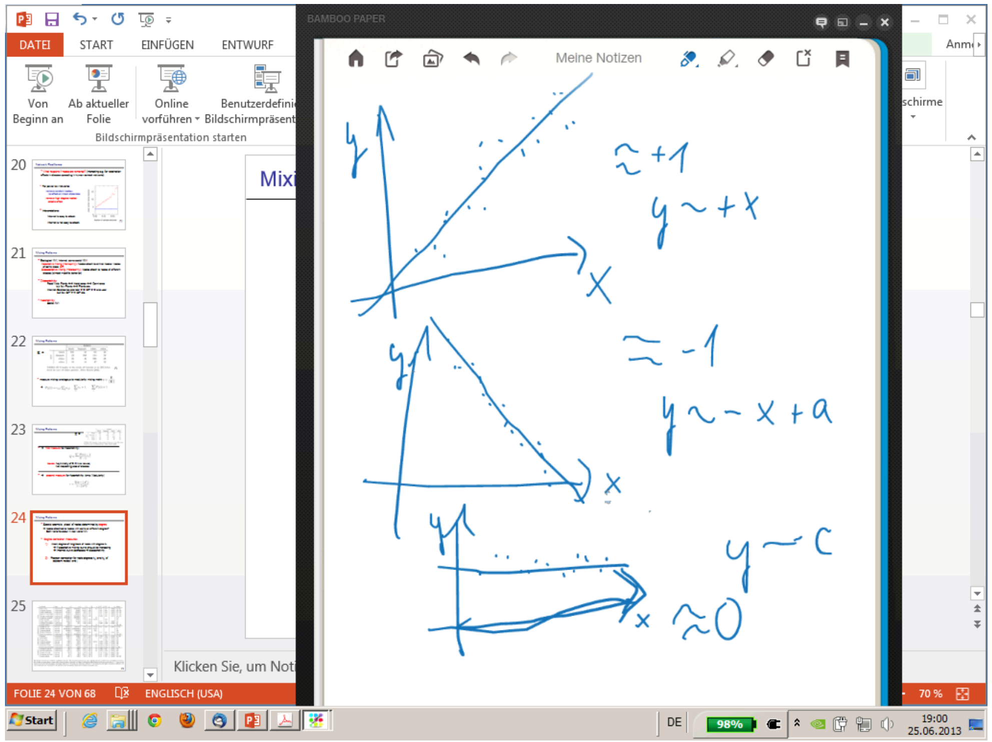
- Special example: „class“ of nodes determined by degree
  → nodes attached to nodes with same or different degree?
  Both variants occur in real world NW

- Degree correlation measures:

  1) mean degree of neighbors of node with degree k:
     → if assortative mixing: curve should be increasing
     → Internet: curve decreases → diassortativity

  2) Pearson correlation for node degrees k_i and k_j of adjacent nodes i and j





	network	type	n	m	z	ℓ	α	$C^{(1)}$	$C^{(2)}$	r	Ref(s).
social	film actors	undirected	449 913	25 516 482	113.43	3.48	2.3	0.20	0.78	0.208	20, 416
	company directors	undirected	7 673	55 392	14.44	4.60	–	0.59	0.88	0.276	105, 323
	math coauthorship	undirected	253 339	496 489	3.92	7.57	–	0.15	0.34	0.120	107, 182
	physics coauthorship	undirected	52 909	245 300	9.27	6.19	–	0.45	0.56	0.363	311, 313
	biology coauthorship	undirected	1 520 251	11 803 064	15.53	4.92	–	0.088	0.60	0.127	311, 313
	telephone call graph	undirected	47 000 000	80 000 000	3.16		2.1				8, 9
	email messages	directed	59 912	86 300	1.44	4.95	1.5/2.0		0.16		136
	email address books	directed	16 881	57 029	3.38	5.22	–	0.17	0.13	0.092	321
	student relationships	undirected	573	477	1.66	16.01	–	0.005	0.001	−0.029	45
	sexual contacts	undirected	2 810				3.2				265, 266
information	WWW nd.edu	directed	269 504	1 497 135	5.55	11.27	2.1/2.4	0.11	0.29	−0.067	14, 34
	WWW Altavista	directed	203 549 046	2 130 000 000	10.46	16.18	2.1/2.7				74
	citation network	directed	783 339	6 716 198	8.57		3.0/–				351
	Roget's Thesaurus	directed	1 022	5 103	4.99	4.87	–	0.13	0.15	0.157	244
	word co-occurrence	undirected	460 902	17 000 000	70.13		2.7		0.44		119, 157
technological	Internet	undirected	10 697	31 992	5.98	3.31	2.5	0.035	0.39	−0.189	86, 148
	power grid	undirected	4 941	6 594	2.67	18.99	–	0.10	0.080	−0.003	416
	train routes	undirected	587	19 603	66.79	2.16	–		0.69	−0.033	366
	software packages	directed	1 439	1 723	1.20	2.42	1.6/1.4	0.070	0.082	−0.016	318
	software classes	directed	1 377	2 213	1.61	1.51	–	0.033	0.012	−0.119	395
	electronic circuits	undirected	24 097	53 248	4.34	11.05	3.0	0.010	0.030	−0.154	155
	peer-to-peer network	undirected	880	1 296	1.47	4.28	2.1	0.012	0.011	−0.366	6, 354
biological	metabolic network	undirected	765	3 686	9.64	2.56	2.2	0.090	0.67	−0.240	214
	protein interactions	undirected	2 115	2 240	2.12	6.80	2.4	0.072	0.071	−0.156	212
	marine food web	directed	135	598	4.43	2.05	–	0.16	0.23	−0.263	204
	freshwater food web	directed	92	997	10.84	1.90	–	0.20	0.087	−0.326	272
	neural network	directed	307	2 359	7.68	3.97	–	0.18	0.28	−0.226	416, 421

BLE II Basic statistics for a number of published networks. The properties measured are: type of graph, directed or undirected; total number of vertices
ber of edges m; mean degree z; mean vertex–vertex distance ℓ; exponent α of degree distribution if the distribution follows a power law (or "–" if not; in/out
onents are given for directed graphs); clustering coefficient $C^{(1)}$ from Eq. (3); clustering coefficient $C^{(2)}$ from Eq. (6); and degree correlation coefficient r, Se
last column gives the citation(s) for the network in the bibliography. Blank entries indicate unavailable data.

[1]

Community and Group Structure

- Is NW well clustered? → see Parts on Clustering

example:
friendship NW
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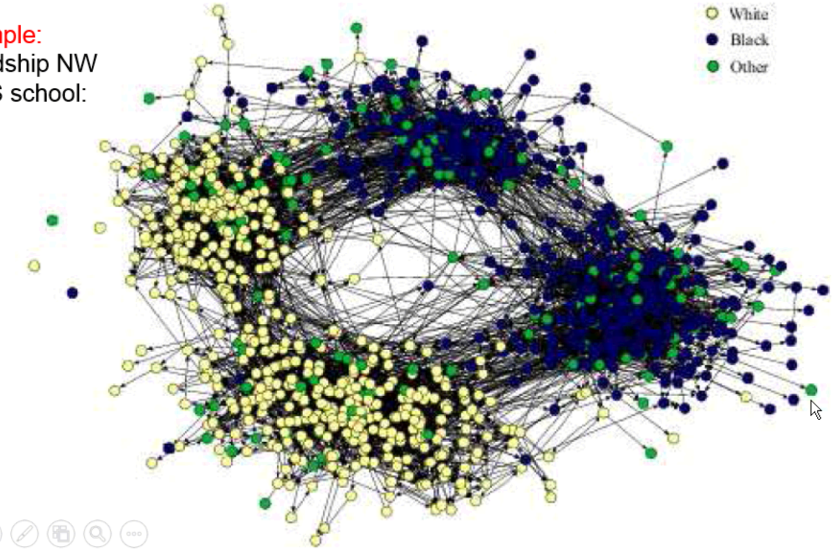

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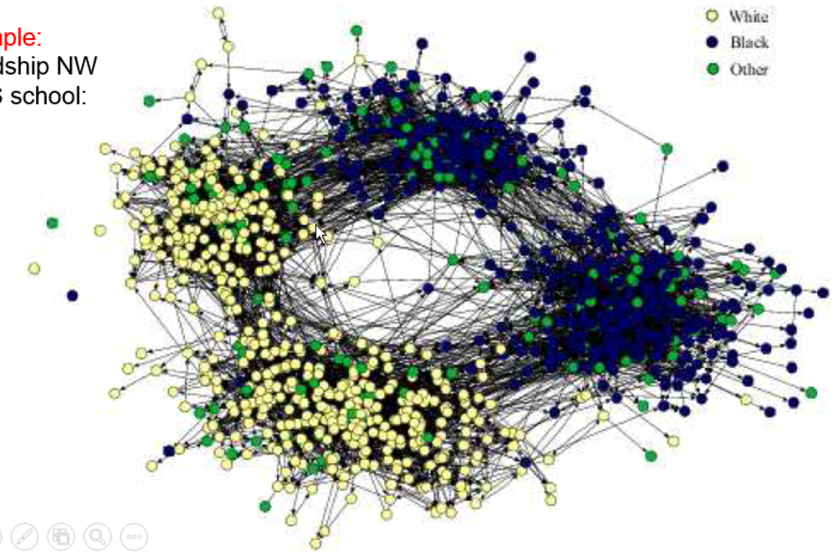

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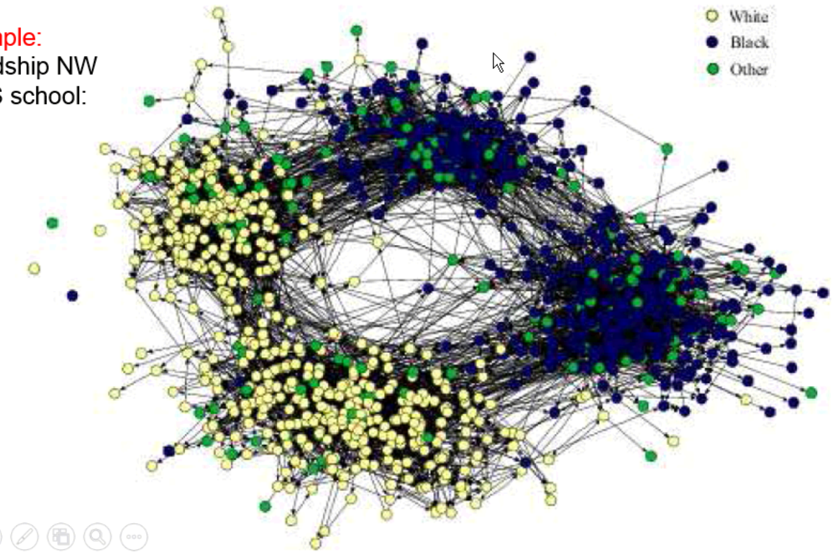

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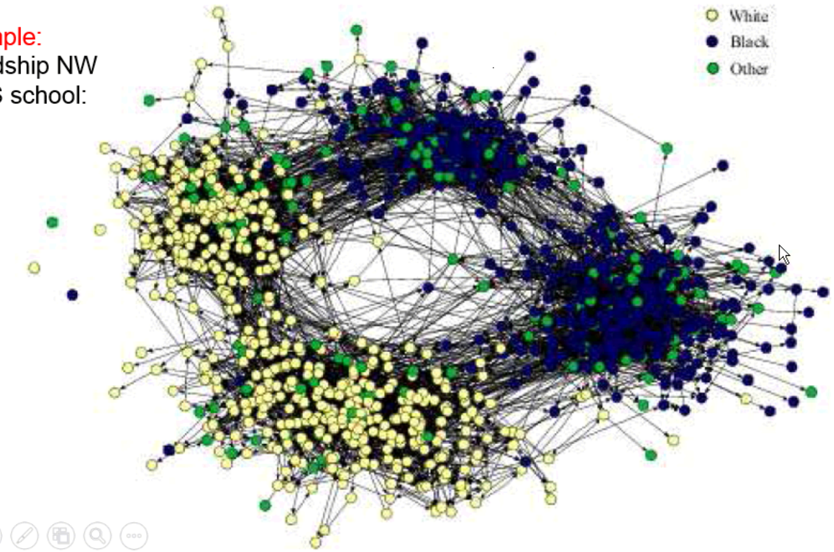

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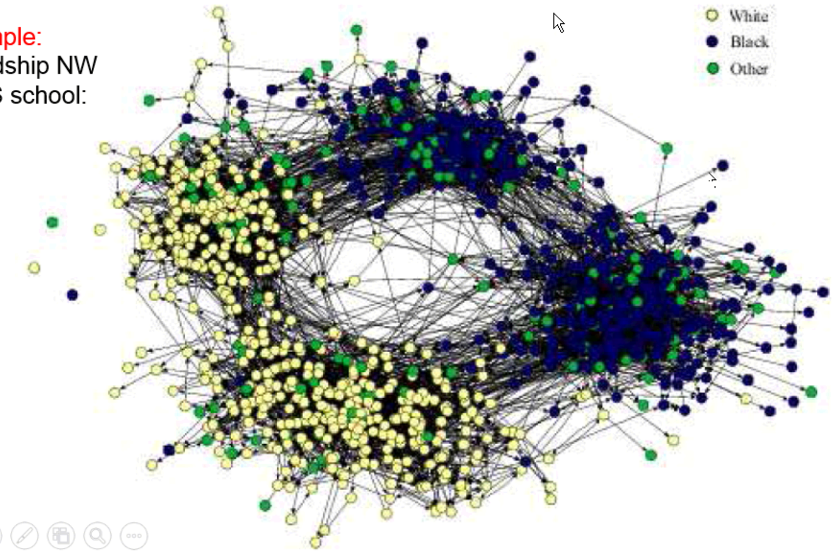

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Random Graph Models: Poisson Graph

- $G_{n,p}$: space of graphs with n nodes and each of the ½ n(n-1) edges appears with probability p

- p_k: probability that a node has degree k:

$$p_k = \binom{n}{k} p^k (1-p)^{n-k} \simeq \frac{z^k e^{-z}}{k!}$$

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„$G_{n,p}$ has property Q with high probability": $P(Q|n,p) \to 1$ iff $n \to \infty$
(adaptated from [2] (which, in turn, is adaptated from [3]))

- In such models $G_{n,p}$ phase transitions exist for properties Q:
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$$\lim_{n\to\infty} P(Q|n,p) = \begin{cases} 0 & \text{if } \lim_{n\to\infty} p(n) / q(n) = 0 \\ 1 & \text{if } \lim_{n\to\infty} p(n) / q(n) = \infty \end{cases}$$

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- Let u be the fraction of nodes that do not belong to giant component X
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- probability for a given node i (with assumed degree k) to be not in X
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- → u (k fixed) == u^k → $u = \sum_{k=0}^{\infty} p_k u^k = e^{-z} \sum_{k=0}^{\infty} \frac{(zu)^k}{k!} = e^{z(u-1)}$

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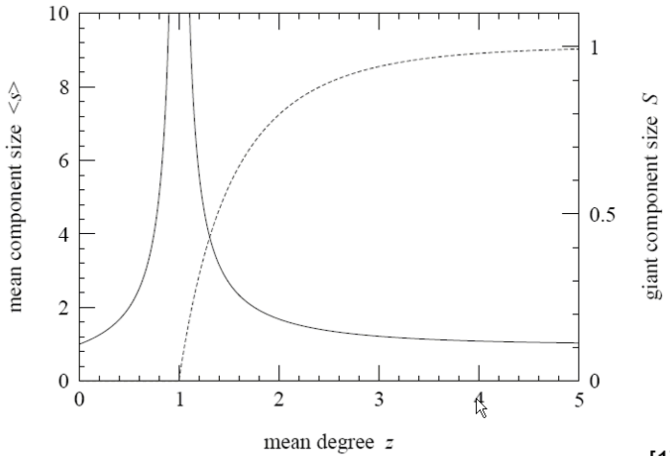

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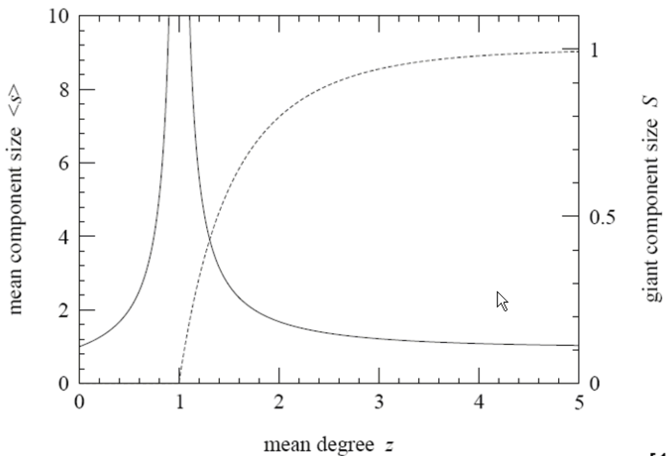

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Very coarse (!!!) estimation of diameter l of $G_{n,p}$:

- average degree of nodes: z
  → in a distance of d from a node i should be approximately z^d many nodes
  → if $z^d = n$: $d = l$
  → $l \sim \log n / \log z \sim \log n$ (if z is kept constant)
- For a more exact derivation of the result see references in [1]
- We see: it is not difficult (in terms of how large must connectivity be) to acheive small diameters

Random Graph Models: Poisson Graph

Unfortunately: small *l* is the only property in congruence with real world NW:

- Clustering coefficient $C^{(1)}$ of $G_{n,p}$:
  - Since $C^{(1)}$ is probability of transitivity and edges are "drawn" independently → $C^{(1)} = p = O(1/n)$ (if z is fixed, as usual)
  - C is usually much larger for real world NW:

	l (real)	*l* (random)	$C^{(2)}$ (real)	C (random)
Film collaboration	3.65	2.99	0.79	0.00027
Power Grid	18.7	12.4	0.08	0.005
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- Degree distribution is Poisson and not power law

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- Furthermore PRG :
  - has random mixing patterns,
  - is not navigatable with local search,
  - has no community structure

Random Graph Models: More Refined Models

- Instead of having connection probability p as in Poisson $G_{n,p}$: demand certain degree distributions p_k (e.g. power law)
- → results are promising but still not in congruence with real world NW
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- Great problem of random graphs: high clustering coeff. / transitivity does not occur for simple models

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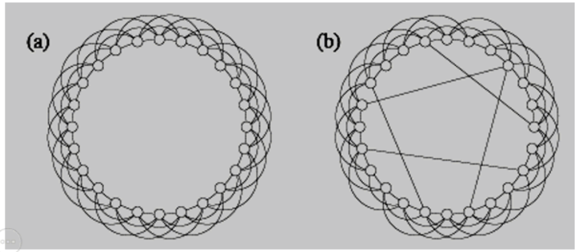
- → Watts & Strogatz 1998: Small World Model
  - L nodes in regular D-dim. lattice + periodic boundary cond.; D=1: Ring
  - each node connected to neighbors in lattice at distance of most k
    → total number of edges = L k
  - „rewiring" of edges with probability p

before rewiring   after rewiring

D = 1; k = 3



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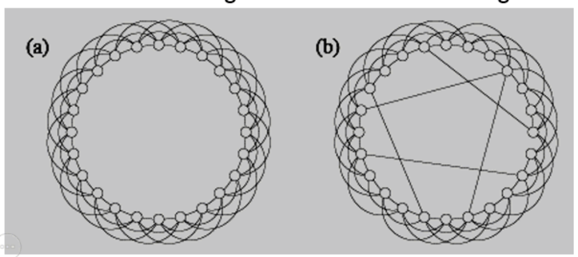

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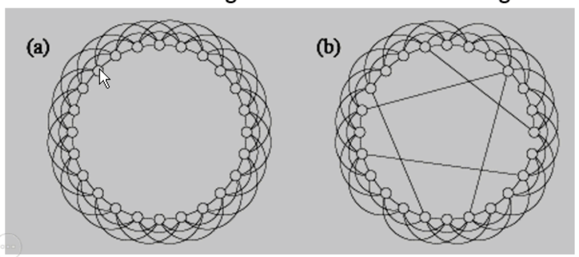

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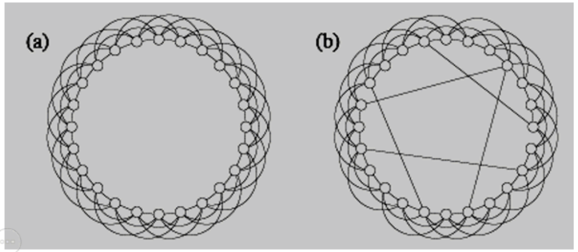

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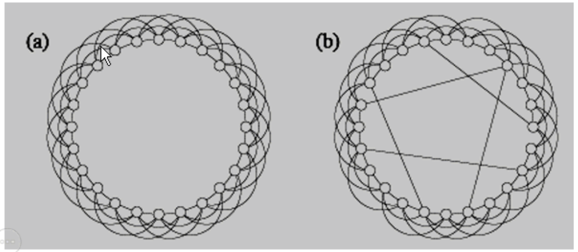
- → Watts & Strogatz 1998: Small World Model
  - L nodes in regular D-dim. lattice + periodic boundary cond.; D=1: Ring
  - each node connected to neighbors in lattice at distance of most k
    → total number of edges = L k
  - „rewiring“ of edges with probability p

before rewiring   after rewiring

D = 1; k = 3



[1]

Watts Strogatz Model

p: transition between regular lattice and sth. like a random graph:
(for D=1:)

- p=0: regular lattice:
  - $C = C^{(1)} = (3k-3)/(4k-2)$ → ¾ for k→∞ → clustering coeff. „ok“
  - l = L / 4k for L→∞ → no small world effect

- p=1: similar to a random graph:
  - C ~ 2k / L for L→∞ → clustering coeff too small
  - l = log L / log k for L→∞ → small world effect.

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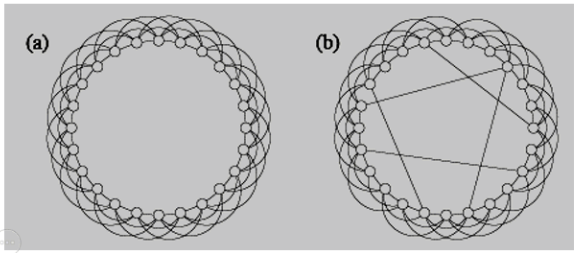

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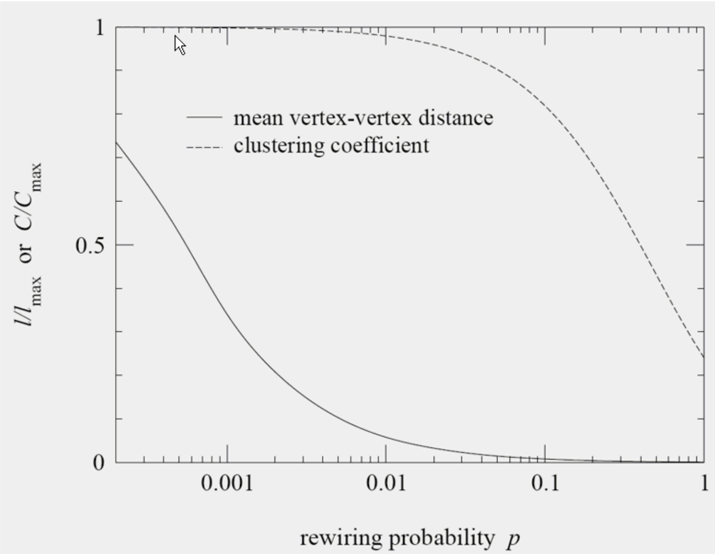
- Interesting area: intermediate values for p: (shown: variant (2), similar in orig. model):



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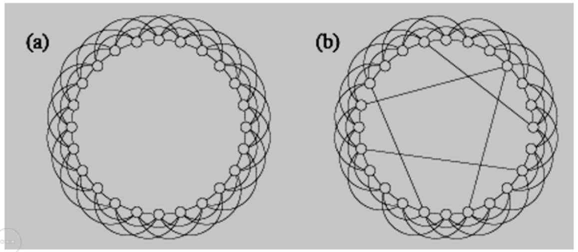

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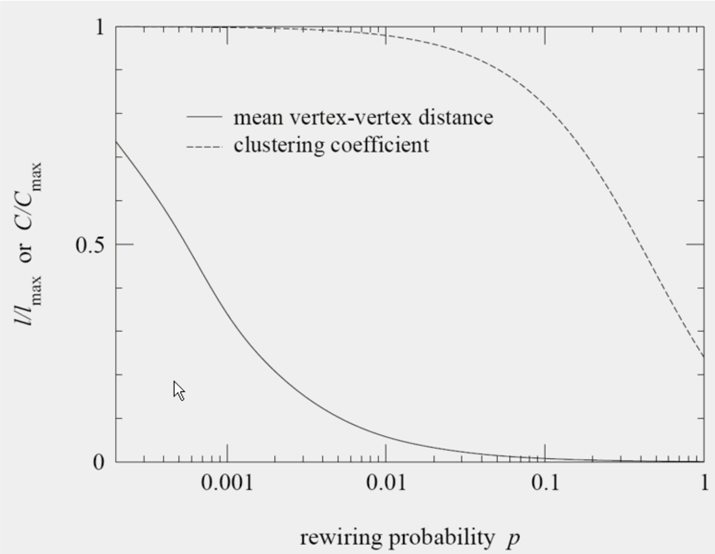

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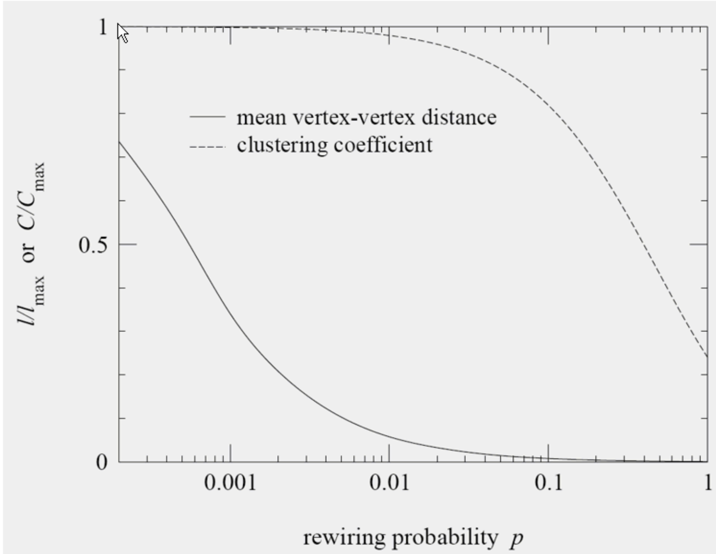

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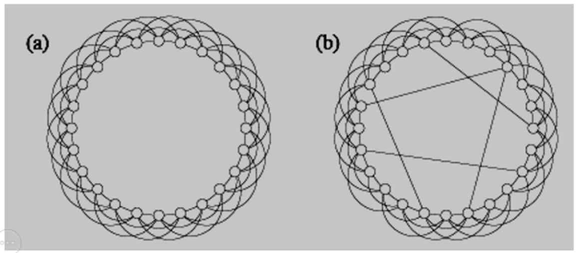

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Watts Strogatz Model

- Variants: -(1)- rewire both „ends" of edges + allow self-edges +....
  → math.easier

  -(2)- only add **additional** shortcut edges (no rewiring)

- For (2):
  - mean total number of shortcuts = L k p
  - mean degree of each node = 2k(1+p)

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