

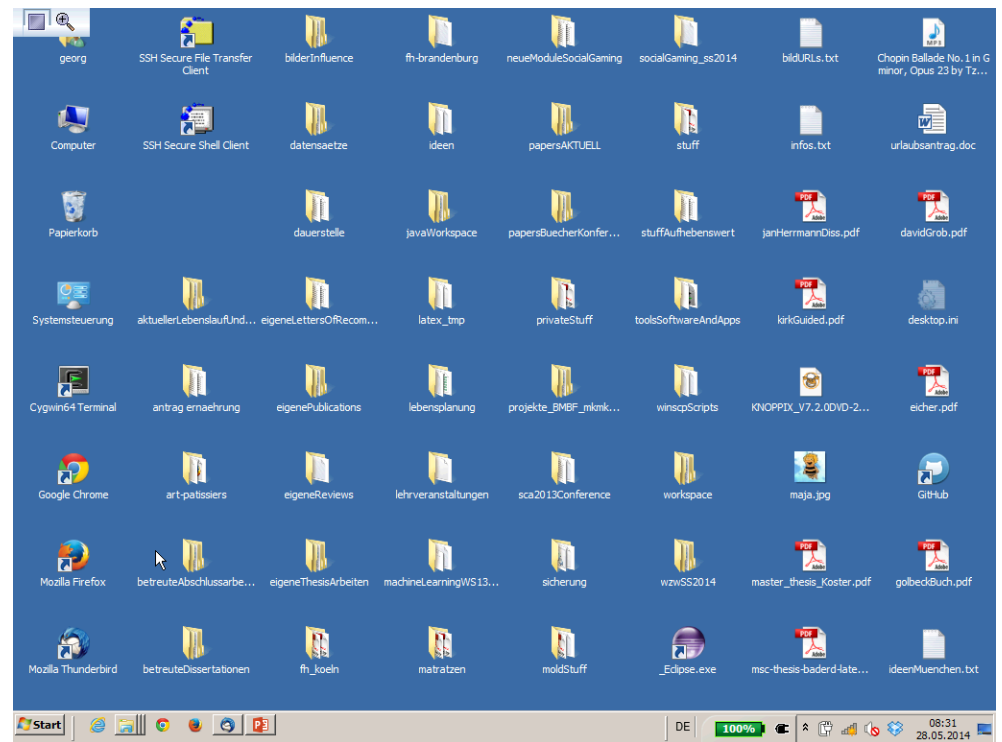
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Title: groh: profile1 (28.05.2014)

Date: Wed May 28 08:32:00 CEST 2014

Duration: 74:41 min

Pages: 77



Real World Networks: Properties
and Models

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Lecture will mostly follow [1], thus
corresponding citations are often omitted to
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Real World Networks: Properties and Models

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Mean Average Path Length

- “Small World Effect”: $l(n)$ “small” $\rightarrow l(n) \in O(\log(n))$
- undirected graph:

$$\ell = \frac{1}{\frac{1}{2}n(n+1)} \sum_{i \geq j} d_{ij}$$

formula also counts 0 distances from i to i : $\frac{1}{2}n(n+1) = \frac{1}{2}n(n-1) + n$

- Expression allowing for disconnected components (where $d_{ij} = \infty$ can occur): harmonic mean:

$$\ell^{-1} = \frac{1}{\frac{1}{2}n(n+1)} \sum_{i \geq j} d_{ij}^{-1}$$



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Transitivity / Clustering Coefficient

- Clustering coefficient (whole graph):

$$C = C^{(1)} = \frac{3 \times \text{number of triangles in the network}}{\text{number of connected triples of vertices}} \quad \rho(\text{FOAF})$$

$$= \frac{6 \times \text{number of triangles in the network}}{\text{number of paths of length two}}$$

- Clustering coefficient (Watts-Strogatz-version, for node i):

$$C_i = \frac{\text{number of triangles connected to vertex } i}{\text{number of triples centered on vertex } i}$$

$$= \frac{|\{e_{\{kj\}} \mid v_k, v_j \in N_i\}|}{\frac{k_i(k_i-1)}{2}} \quad (\text{see Introduction, } k_i = \text{degree of node } i)$$

Clustering coefficient (Watts-Strogatz-version, for whole graph):

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mean of ratio instead of ratio of means

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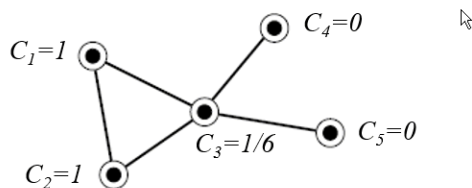
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Transitivity / Clustering Coefficient

Example:

$$C^{(1)} = \frac{3 \times \text{number of triangles in the network}}{\text{number of connected triples of vertices}} = \frac{3 \times 1}{8} = 0.375$$



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$$C^{(2)} = 1/5 (1 + 1 + 1/6 + 0 + 0) = 13/30 = 0.433333$$

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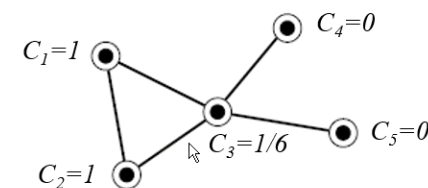
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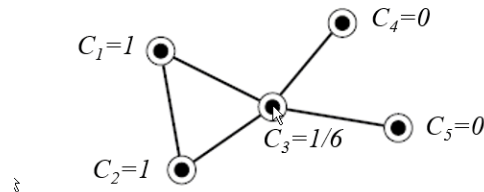
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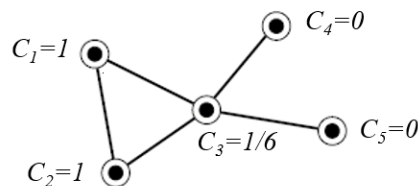
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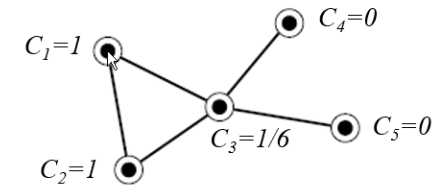
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Transitivity / Clustering Coefficient

- Evidence → for many real world NW:

$$C^{(1)} \sim O(1) \quad \text{for } n \rightarrow \infty$$

- For random NW:

$$C^{(1)}, C^{(2)} \sim O(1/n) \quad \text{for } n \rightarrow \infty$$



	network	type	n	m	z	ℓ	α	$C^{(1)}$	$C^{(2)}$	r	Ref(s).
social	film actors	undirected	449913	25516482	113.43	3.48	2.3	0.20	0.78	0.208	20, 416
	company directors	undirected	7673	55392	14.44	4.60	-	0.59	0.88	0.276	105, 323
	math coauthorship	undirected	253339	496489	3.92	7.57	-	0.15	0.34	0.120	107, 182
	physics coauthorship	undirected	52909	245300	9.27	6.19	-	0.45	0.56	0.363	311, 313
	biology coauthorship	undirected	1520251	11803064	15.53	4.92	-	0.088	0.60	0.127	311, 313
	telephone call graph	undirected	47000000	80000000	3.16		2.1				8, 9
	email messages	directed	59912	86300	1.44	4.95	1.5/2.0		0.16		136
	email address books	directed	16881	57029	3.38	5.22	-	0.17	0.13	0.092	321
	student relationships	undirected	573	477	1.66	16.01	-	0.005	0.001	-0.029	45
	sexual contacts	undirected	2810				3.2				265, 266
information	WWW nd.edu	directed	269504	1497135	5.55	11.27	2.1/2.4	0.11	0.29	-0.067	14, 34
	WWW Altavista	directed	203549046	2130000000	10.46	16.18	2.1/2.7				74
	citation network	directed	783339	6716198	8.57		3.0/-				351
	Roget's Thesaurus	directed	1022	5103	4.99	4.87	-	0.13	0.15	0.157	244
	word co-occurrence	undirected	460902	17000000	70.13		2.7		0.44		119, 157
technological	Internet	undirected	10697	31992	5.98	3.31	2.5	0.035	0.39	-0.189	86, 148
	power grid	undirected	4941	6594	2.67	18.99	-	0.10	0.080	-0.003	416
	train routes	undirected	587	19603	66.79	2.16	-	0.69	-0.033	0.366	
	software packages	directed	1439	1723	1.20	2.42	1.6/1.4	0.070	0.082	-0.016	318
	software classes	directed	1377	2213	1.61	1.51	-	0.033	0.012	-0.119	395
	electronic circuits	undirected	24097	53248	4.34	11.05	3.0	0.010	0.030	-0.154	155
	peer-to-peer network	undirected	880	1296	1.47	4.28	2.1	0.012	0.011	-0.366	6, 354
biological	metabolic network	undirected	765	3686	9.64	2.56	2.2	0.090	0.67	-0.240	214
	protein interactions	undirected	2115	2240	2.12	6.80	2.4	0.072	0.071	-0.156	212
	marine food web	directed	135	598	4.43	2.05	-	0.16	0.23	-0.263	204
	freshwater food web	directed	92	997	10.84	1.90	-	0.20	0.087	-0.326	272
	neural network	directed	307	2359	7.68	3.97	-	0.18	0.28	-0.226	416, 421

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- **Notation:**

$p(k) = p_k =$ fraction of nodes having degree k

- **Cumulative distribution:**

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- **power law:**

$$p_k \sim k^{-\alpha}$$

$$\rightarrow P_k \sim \sum_{k'=k}^{\infty} k'^{-\alpha} \sim k^{-(\alpha-1)}$$

- **exponential:**

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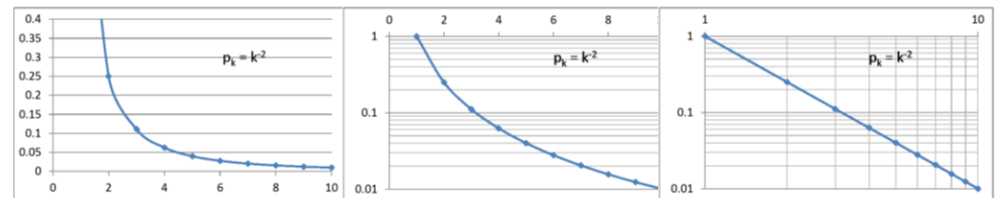
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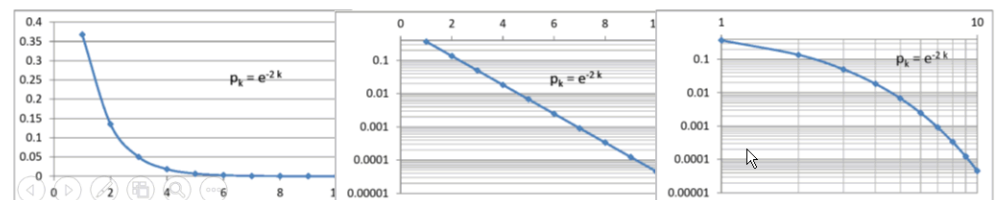
$$\rightarrow P_k = \sum_{k'=k}^{\infty} p_{k'} \sim \sum_{k'=k}^{\infty} e^{-k'/\kappa} \sim e^{-k/\kappa}$$

Degree Distribution

$$p_k \sim k^{-\alpha}$$

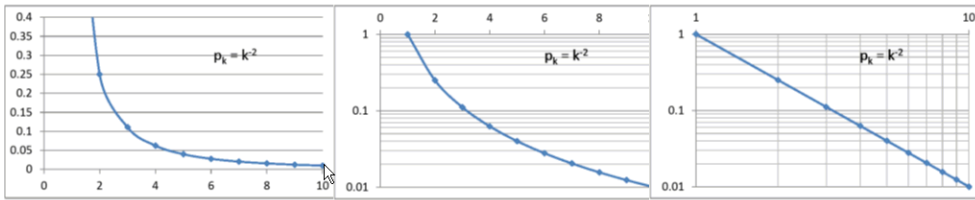


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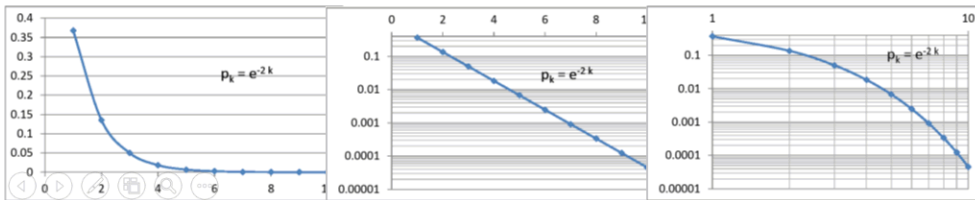


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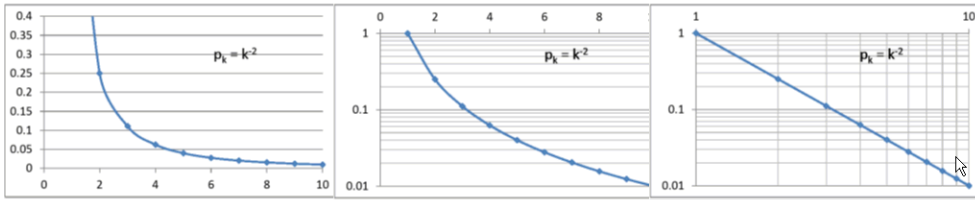


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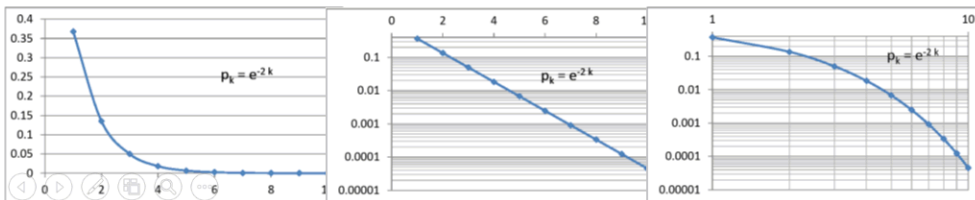


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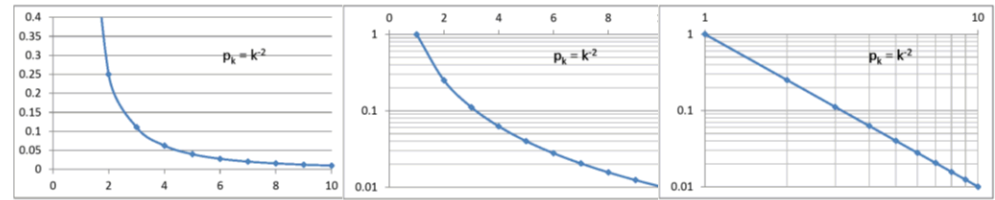


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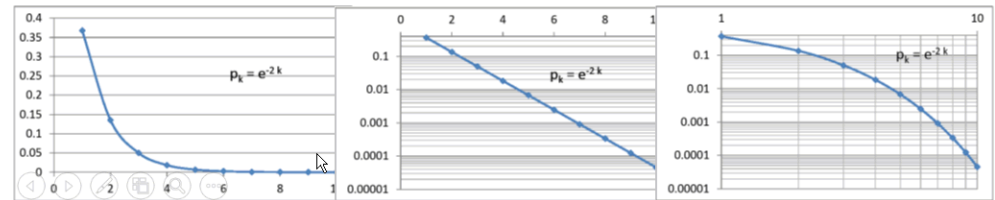


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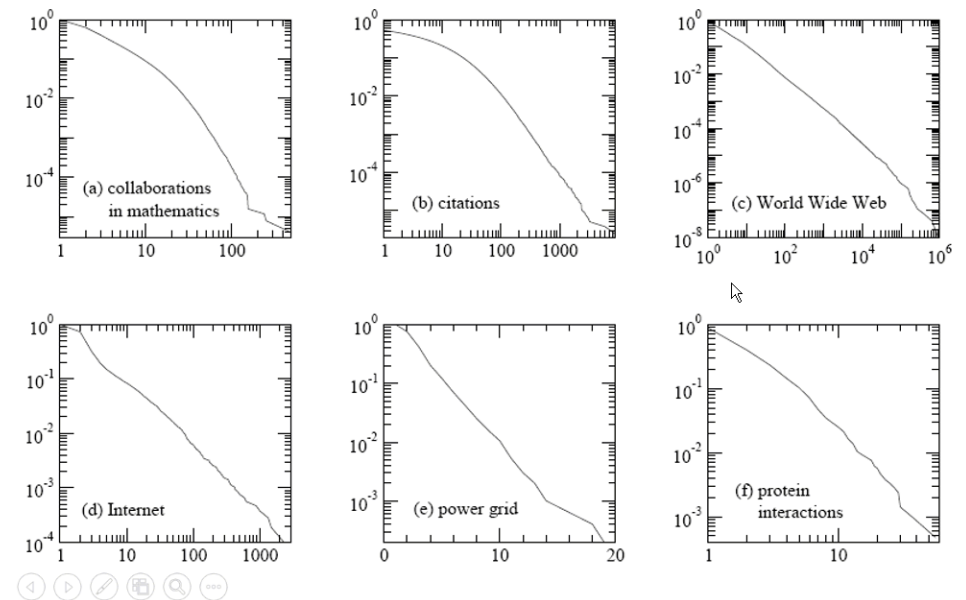


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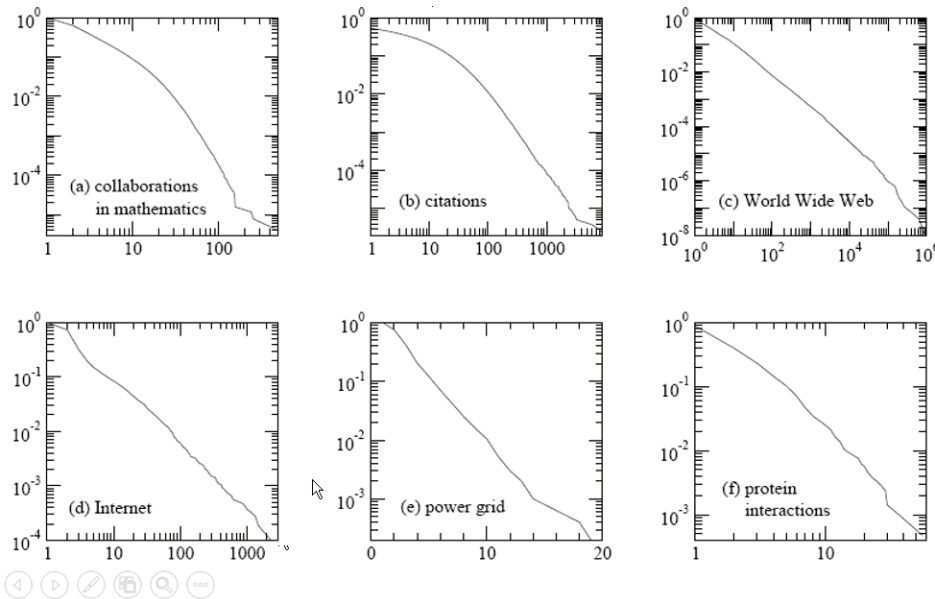
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Cumulative distributions P_k of example real world NW



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Cumulative distributions P_k of example real world NW



[1]

Degree Distribution

“Power law” == “Scale free”:

- $f(x) = x^\alpha$ is only solution to functional equation formalizing scale freedom $f(ax) = b f(x)$

- in other words: change of scale \rightarrow f still „looks the same“

- other point of view:

Although we can compute the expectation $E(k) = \sum_k k k^{-\alpha}$ if $\alpha > 1$, the **variance** (error bars) $\text{Var}(k) = \sum_k (k - E(k))^2 k^{-\alpha}$ **diverges** \rightarrow we „cannot be shure about k“ \rightarrow „no characteristic scale“ \rightarrow „scale free“

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Examples:

- **Power law:** citation NW, WWW, Internet, metabolic NW, telephone call NW, human sexual contact NW etc.
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Maximum Degree

- „less or equal than one vertex with k_{\max} “
→ $np_{k_{\max}} = 1$ → for power law $p_k = k^{-\alpha}$: $k_{\max} \sim n^{1/\alpha}$
but: not very accurate estimation

- **Other estimation:**

- prob p of „exactly m nodes with k and rest of nodes smaller than k“:

$$\binom{n}{m} p_k^m (1 - P_k)^{n-m}$$

- → prob of k being the highest degree in graph:

$$\begin{aligned} h_k &= \sum_{m=1}^n \binom{n}{m} p_k^m (1 - P_k)^{n-m} \\ &= (p_k + 1 - P_k)^n - (1 - P_k)^n \end{aligned}$$

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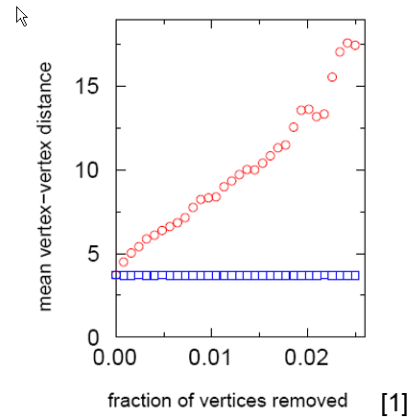
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- What happens if nodes are removed? (interesting e.g. for vaccination effects in disease spreading in human contact networks)

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 - remove random nodes :
no effect on mean distances
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drastic effect

- Interpretations:
 - Internet is easy to attack
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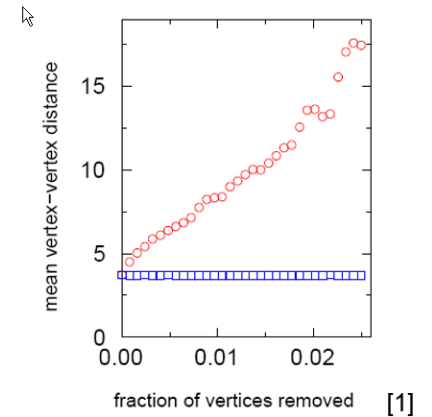


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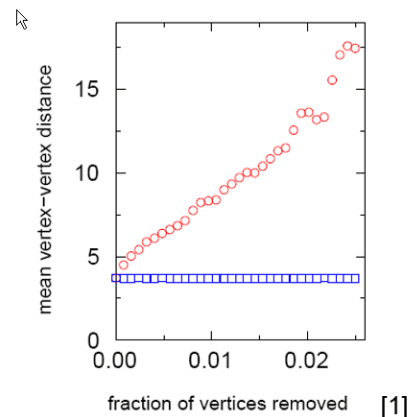


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- Ecological NW, Internet, some social NW:
 - Assortative Mixing (Homophily): Nodes attach to similar nodes / nodes of same class OR
 - Disassortative Mixing (Heterophily): Nodes attach to nodes of different classes (almost n-partite behavior)

- Diassortativity:
 - Food Web: Plants \leftrightarrow Herbivores \leftrightarrow Carnivores
but few Plants \leftrightarrow Plants etc.
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men	black	506	32	69	26
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	white	26	46	599	68
	other	10	14	47	32

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issues: Asymmetry of E \rightarrow two values;
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 Both variants occur in real world NW

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	network	type	n	m	z	ℓ	α	$C^{(1)}$	$C^{(2)}$	r	Ref(s).
social	film actors	undirected	449913	25 516 482	113.43	3.48	2.3	0.20	0.78	0.208	20, 416
	company directors	undirected	7 673	55 392	14.44	4.60	-	0.59	0.88	0.276	105, 323
	math coauthorship	undirected	253 339	496 489	3.92	7.57	-	0.15	0.34	0.120	107, 182
	physics coauthorship	undirected	52 909	245 300	9.27	6.19	-	0.45	0.56	0.363	311, 313
	biology coauthorship	undirected	1 520 251	11 803 064	15.53	4.92	-	0.088	0.60	0.127	311, 313
	telephone call graph	undirected	47 000 000	80 000 000	3.16	-	2.1	-	-	-	8, 9
	email messages	directed	59 912	86 300	1.44	4.95	1.5/2.0	-	0.16	-	136
	email address books	directed	16 881	57 029	3.38	5.22	-	0.17	0.13	0.092	321
	student relationships	undirected	573	477	1.66	16.01	-	0.005	0.001	-0.029	45
	sexual contacts	undirected	2 810	-	-	-	3.2	-	-	-	265, 266
information	WWW nd.edu	directed	269 504	1 497 135	5.55	11.27	2.1/2.4	0.11	0.29	-0.067	14, 34
	WWW Altavista	directed	203 549 046	2 130 000 000	10.46	16.18	2.1/2.7	-	-	-	74
	citation network	directed	783 339	6 716 198	8.57	-	3.0/-	-	-	-	351
	Roget's Thesaurus	directed	1 022	5 103	4.99	4.87	-	0.13	0.15	0.157	244
	word co-occurrence	undirected	460 902	17 000 000	70.13	-	2.7	-	0.44	-	119, 157
technological	Internet	undirected	10 697	31 992	5.98	3.31	2.5	0.035	0.39	-0.189	86, 148
	power grid	undirected	4 941	6 594	2.67	18.99	-	0.10	0.080	-0.003	416
	train routes	undirected	587	19 603	66.79	2.16	-	0.69	-0.033	0.366	366
	software packages	directed	1 439	1 723	1.20	2.42	1.6/1.4	0.070	0.082	-0.016	318
	software classes	directed	1 377	2 213	1.61	1.51	-	0.033	0.012	-0.119	395
	electronic circuits	undirected	24 097	53 248	4.34	11.05	3.0	0.010	0.030	-0.154	155
	peer-to-peer network	undirected	880	1 296	1.47	4.28	2.1	0.012	0.011	-0.366	6, 354
biological	metabolic network	undirected	765	3 686	9.64	2.56	2.2	0.090	0.67	-0.240	214
	protein interactions	undirected	2 115	2 240	2.12	6.80	2.4	0.072	0.071	-0.156	212
	marine food web	directed	135	598	4.43	2.05	-	0.16	0.23	-0.263	204
	freshwater food web	directed	92	997	10.84	1.90	-	0.20	0.087	-0.326	272
	neural network	directed	307	2 359	7.68	3.97	-	0.18	0.28	-0.226	416, 421



	network	type	n	m	z	ℓ	α	$C^{(1)}$	$C^{(2)}$	r	Ref(s).
social	film actors	undirected	449913	25 516 482	113.43	3.48	2.3	0.20	0.78	0.208	20, 416
	company directors	undirected	7 673	55 392	14.44	4.60	-	0.59	0.88	0.276	105, 323
	math coauthorship	undirected	253 339	496 489	3.92	7.57	-	0.15	0.34	0.120	107, 182
	physics coauthorship	undirected	52 909	245 300	9.27	6.19	-	0.45	0.56	0.363	311, 313
	biology coauthorship	undirected	1 520 251	11 803 064	15.53	4.92	-	0.088	0.60	0.127	311, 313
	telephone call graph	undirected	47 000 000	80 000 000	3.16	-	2.1	-	-	-	8, 9
	email messages	directed	59 912	86 300	1.44	4.95	1.5/2.0	-	0.16	-	136
	email address books	directed	16 881	57 029	3.38	5.22	-	0.17	0.13	0.092	321
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	protein interactions	undirected	2 115	2 240	2.12	6.80	2.4	0.072	0.071	-0.156	212
	marine food web	directed	135	598	4.43	2.05	-	0.16	0.23	-0.263	204
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[1]

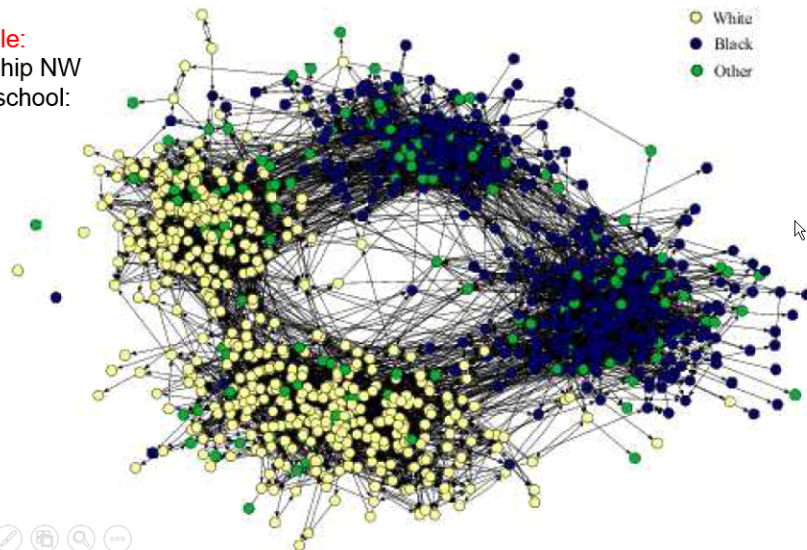


[1]

Community and Group Structure

- Is NW well clustered? → see Parts on Clustering

example:
friendship NW
in US school:



Navigability of NW

- Milgram showed: short paths exist
BUT: How do people find them?

→ see Part „Social Networks in Time and Space“

Component Structure

- Does a giant component exist?

→ see section on random graphs



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